On the Speed-Accuracy Trade-off in Collective Decision Making

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***Abstract*— We study collective decision making in human groups performing a two alternative choice task. We model the evidence aggregation process across the network using a coupled drift diffusion model (DDM) and consider the free response paradigm in which humans take their time to make the decision. We analyze the coupled DDM under a mean-field type approximation and characterize approximate error rates and expected decision times for each individual in the group as a function of their location in the network. We also provide approximations to the first passage time distributions for each individual. We elucidate on criteria to select thresholds for decision making in human groups as well as in engineering applications.**

I. INTRODUCTION

Recent years have witnessed a considerable interest in rigorous understanding of a group's wisdom and associated decision making process. Consequently, there has been extensive research that has led to several models for social network dynamics [1], [2]. One of the fundamental drawbacks of such models is that they do not capture the psychophysics of individuals in the group and thus become ineffective in applications involving real-time evolution of human psychophysical state. One such application is the deployment of a team of human operators that supervises the operation of the automaton in complex and uncertain environments. Such operators collect information from the environment, interact with each other and communicate their beliefs on the state of the environment. In such systems, efficient models for the evolution of each individual's psychophysical state and associated decision making process are fundamental to design of effective human-automaton teams.

In this paper, we focus on the speed-accuracy trade-off in collective decision making using the context of two alternative choice problems. The two alternative choice problem is a simplification of many decision making scenarios and captures the essence of the speed-accuracy trade-off in a variety of situations encountered by animal groups [3], [4]. Moreover, the human performance in two alternative choice tasks is extensively studied and well understood [5], [6], [7]. In particular, the human performance in a two alternative choice task is well modeled by a *drift-diffusion model* (DDM) and its variants. Furthermore, these variants of the DDM under the optimal choice of their parameters are equivalent to the DDM [5].

Collective decision making in animal and human groups has fetched significant interest in a broad scientific community [8], [9], [10]. The collective decision making in human groups is typically studied under two extreme communication regimes: the so-called *ideal group* and the *Condorcet group*. In an ideal group, each decision maker interacts with every other decision maker and the group arrives at a consensus decision, while in a Condorcet group, decision makers do not interact with each other and a majority rule is employed to reach a decision. Collective decision making in ideal human groups and Condorcet human groups is studied in [10] using the classical signal detection model for human performance in two alternative choice tasks. Human decision making is typically studied under two paradigms, namely, *interrogation*, and *free response*. In the interrogation paradigm, the human has to make a decision at the end of a prescribed time duration, while in the free response paradigm, the human takes their time to make a decision. Collective decision making in Condorcet human groups using the DDM and free response paradigm is studied in [11], [12]. Collective decision making in ideal human groups using the DDM and interrogation paradigm is studied in [13]. Related collective decision making models in animal groups are studied in [14]. In this paper, we study collective decision making in ideal groups using the DDM and free response paradigm.

The DDM is a continuous time approximation to the evidence aggregation process in a hypothesis testing problem. Moreover, the finite sample and the sequential hypothesis testing problems correspond to the interrogation and the free response paradigm in human decision making, respectively. Consequently, the collective decision making problem in human groups is similar to distributed hypothesis testing problems studied in the engineering literature [15], [16], [17]. In particular, Braca *et al.* [16] study distributed implementations of the finite sample as well as the sequential hypothesis testing problems. They use a *running consensus* algorithm to aggregate the test statistic across the network and show that the proposed algorithm achieves the performance of a centralized algorithm asymptotically. In contrast to [16], we rely on the Laplacian flow [18] to aggregate evidence across network. Moreover, we approximately characterize the finite time behavior of the coupled DDM under the free response paradigm.

The contributions of this paper are fourfold. First, we use the Laplacian flow based evidence aggregation model for human groups [13] in conjugation with a mean-field type approximation to determine an effective time varying DDM for the evolution of each individual's evidence as a function of their location in the network. Second, we characterize lower and upper bounds on the error rates and decision times associated with the effective DDM. We show that the upper bound on the error rate and the lower bound on the

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expected decision time are asymptotically achieved. Third, we determine lower and upper bounds on the first passage time distribution for the effective DDM. Fourth and last, we elucidate on various threshold selection criteria, namely, the Wald-like criterion, Bayes criterion, and reward rate criterion.

The remainder of the paper is organized as follows. We review decision making models for human and human groups in Section II. We determine the effective DDM for each individual in the group using a mean-field type approximation and characterize its properties in Section III. We elucidate on the concepts developed in the paper through some examples in Section IV. Our conclusions are presented in Section V.

II. Human Decision Making Models

In this section, we present the DDM and the coupled DDM that model the evidence aggregation process in two alternative choice tasks for a single human and a human group, respectively.

A. Drift Diffusion Model

A two alternative choice task [5] is a decision making scenario in which a person has to chose between two plausible alternatives. In a two alternative choice task, the difference between the likelihood of each alternative (evidence) is aggregated and the aggregated evidence is compared against a threshold to make a decision. The decision making is studied under two paradigms, namely, *interrogation* and *free-response*. In the interrogation paradigm, a time duration is prescribed to the human who decides on an alternative at the end of this duration. In particular, by the end of the prescribed duration, the human compares the aggregated evidence against a single threshold, and chooses an alternative. In the free response paradigm, the human subject is free to take as much time as needed to make a reliable decision. In this paradigm, the human compares the aggregated evidence against two thresholds and decides on an alternative only if the associated threshold is crossed; otherwise, the human aggregates more evidence. The evidence aggregation is well modeled by the drift-diffusion process [5] defined by

$$dx(t) = \beta dt + \sigma dW(t), \quad x(0) = x_0, \tag{1}$$

where $\beta \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{>0}$ are, respectively, the drift rate and the diffusion rate, $W(t)$ is the standard one dimensional Wiener process, $x(t)$ is the aggregate evidence at time t, and x_0 is the initial evidence (see [5] for the details of the model).

In this paper, we study decision making under the free response paradigm, which is modeled in the following way. At each time $\tau \in \mathbb{R}_{\geq 0}$, the human compares the aggregated evidence against two symmetrically chosen thresholds $\pm\eta, \eta \in \mathbb{R}_{\geq 0}$. In particular, if $x(\tau) \geq \eta$, then the human decides in favor of the first alternative; if $x(\tau) \leq -\eta$, then the human decides in favor of the second alternative; otherwise, the human collects more evidence.

B. Coupled drift diffusion model

Consider a set of n decision makers performing a two alternative choice task and let their interaction topology be modeled by a connected undirected graph $\mathcal{G}$ with Laplacian matrix $L \in \mathbb{R}^{n \times n}$. The evidence aggregation in collective decision making is modeled in the following way. At each time $t \in \mathbb{R}_{\geq 0}$, every decision maker $k \in \{1, \dots, n\}$ (i) computes a convex combination of her evidence and her neighbor's evidence; (ii) collects a new evidence; and (iii) adds the new evidence to the convex combination. This collective evidence aggregation process is mathematically described by the following coupled drift diffusion model [13]:

$$d\boldsymbol{x}(t) = \big(\beta \mathbf{1}_n - L\boldsymbol{x}(t)\big)dt + \sigma^2 d\boldsymbol{W}(t), \tag{2}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the vector of evidence aggregated by decision makers until time t, $\boldsymbol{W}(t) \in \mathbb{R}^n$ is the vector of n independent standard Weiner processes, and $\mathbf{1}_n$ is the column n-vector of all ones. It should be noted that the coupled DDM (2) captures the interaction among individuals using the Laplacian flow dynamics. The Laplacian flow is the continuous time equivalent of the classical DeGroot model [2], [19] that captures the consensus seeking process in human groups.

The solution to the system (2) is a Gaussian process, and for $\boldsymbol{x}(0) = \mathbf{0}_n$, where $\mathbf{0}_n$ is the n-vector of all zeros,

$$\begin{aligned} \mathbb{E}[x(t)] &= \beta t \mathbf{1}_n, \\ \text{Cov}(x_k(t), x_j(t)) &= \frac{\sigma^2 t}{n} + \sigma^2 \sum_{p=2}^{n} \frac{1 - e^{-2\lambda_p t}}{2\lambda_p} u_k^{(p)} u_j^{(p)}, \end{aligned} \tag{3}$$

for $k, j \in \{1, \dots, n\}$, where $\lambda_p,\ p \in \{2, \dots, n\}$, are non-zero eigenvalues of the Laplacian matrix, and $u_k^{(p)}$ is the k-th component of the normalized eigenvector associated with eigenvalue λ_p (see [13] for details).

*Remark 1 (**Generalized ideal group**):* In contrast to the standard ideal group analysis [10] that assumes each individual interacts with every other individual, in (2) each individual interacts only with its neighbors in the interaction graph $\mathcal{G}$. Thus, the coupled DDM (2) generalizes the ideal group model and captures more general interactions, e.g., organizational hierarchies. □

III. Coupled DDM: Free Response Paradigm

In this section, we characterize the performance of each decision maker in the network under the free response paradigm. We first present a mean-field type approximation to determine an effective DDM for each decision maker. We then characterize error rates, decision times, and first passage time distributions associated with the effective DDM. We close this section with a discussion on threshold selection criteria. We study the free response paradigm under the following assumption:

*Assumption 1 (**Persistent Evidence Aggregation**):* Each decision maker continues to aggregate and communicate evidence according to the coupled DDM (2) even after reaching a decision. □

A. Effective DDM at each node

The free response paradigm for the coupled DDM correspond to the boundary crossing of the n-dimensional Weiner process. In general, for $n > 1$, boundary crossing properties of the Weiner process are hard to characterize analytically, and a few available analytic solutions do not provide much insight into the properties. Therefore, we resort to approximations for the coupled DDM. We note that, at each time t, the coupled DDM is a probabilistic graphical model [20] in which a generic node k corresponds to the random variable $x_k(t)$. The mean-field approximation to a probabilistic graphical model approximates the coupled joint distribution of all the random variables with a joint distribution that factorizes over each random variable and is close to the coupled joint distribution in an appropriate sense [20].

In a similar spirit, we approximate the coupled DDM with n independent effective DDMs such that, at any time t, the distribution of the evidence for the k-th effective DDM is the same as the marginal distribution of $x_k(t)$ in the coupled DDM. The coupled DDM captures the evidence aggregation by any decision maker as a Gaussian process. It follows from equation (3) that the evidence aggregated by the k-th decision maker until time t is marginally distributed according to a normal distribution with mean βt and variance $\frac{\sigma^2 t}{n} + \sigma^2 \sum_{p=2}^{n} \frac{1-e^{-2\lambda_p t}}{2\lambda_p} {u_k^{(p)}}^2$. Accordingly, the evidence aggregation by the k-th decision maker is approximated by the following effective drift diffusion model:

$$dx_k(t) = \beta dt + \sigma_k(t) dW(t), \tag{4}$$

where $\sigma_k(t) = \sigma\sqrt{\frac{1}{n} + \sum_{p=2}^{n} e^{-2\lambda_p t} {u_k^{(p)}}^2}$. The effective DDM (4) captures the evolution of the evidence for the k-th individual in the interrogation paradigm. In the spirit of [21], we use the interrogation paradigm model (4) to approximate the free response paradigm for the coupled DDM (2). Our analysis of the free response paradigm for the effective DDM (4) is similar to the standard martingale based analysis for the standard DDM [22].

We now introduce some notation. Let $\mathcal{F}_\tau^k, \tau \in \mathbb{R}_{\geq 0}$ be the sigma algebra generated by $\{x_k(t)\}_{t\in[0,\tau]}$. Let decision time T_k, measurable with respect to the filtration $\mathcal{F}_\tau^k, \tau \in \mathbb{R}_{\geq 0}$, be defined by $T_k = \inf\{\tau \in \mathbb{R}_{\geq 0} \cup \{+\infty\} \mid x_k(\tau) \in \{-\eta_k, +\eta_k\}\}$, where $\eta_k \in \mathbb{R}_{>0}$ is the threshold for the k-th individual.

B. Error rates and decision times

The error rate is the probability that the human decides in favor of an incorrect alternative, and the decision time is the expected time the human takes to decide on an alternative. If $\beta > 0$ ($\beta < 0$), then an erroneous decision is made if the evidence crosses the threshold $-\eta_k$ ($+\eta_k$) before crossing $+\eta_k$ ($-\eta_k$). Without loss of generality, we assume that $\beta > 0$. We denote the error rate for k-th individual by $\mathtt{ER}_k$. We now determine the error rates and the expected decision times for the free response paradigm associated with effective DDM (4). We first introduce some notation. Define $\mu_k = \big(\sigma^2 \sum_{p=2}^{n} \frac{1}{2\lambda_p} {u_k^{(p)}}^2\big)^{-1}$ for each $k \in \{1,\dots,n\}$. Note that μ_k is called the *node certainty index* and is a measure of the accuracy of individual k [13]. Let the variance of the k-th individual at time t be defined by $\varsigma_k^2(t) = \frac{\sigma^2 t}{n} + \sigma^2 \sum_{p=2}^{n} \frac{1-e^{-2\lambda_p t}}{2\lambda_p} {u_k^{(p)}}^2$. Let the stochastic process $\{y_k(t)\}_{t\in\mathbb{R}_{\geq 0}}$ be defined by

$$y_k(t) = \exp\Big(\theta x_k(t) - \theta\beta t - \frac{1}{2}\theta^2\sigma^2\Big(\frac{t}{n} - \sum_{p=2}^{n} \frac{e^{-2\lambda_p t}}{2\lambda_p} {u_k^{(p)}}^2\Big)\Big),$$

for some $\theta \in \mathbb{R}$ and $k \in \{1,\dots,n\}$. We will show that the stochastic process $\{y_k(t)\}_{t\in\mathbb{R}_{\geq 0}}$ is a martingale and utilize it to determine the error rates. We now state the following theorem about error rates and expected decision times.

*Theorem 1 (**Error Rates and Decision Times**):* For the effective DDM (4) and the free response decision making paradigm, the following statements hold:

(i) the stochastic process $\{y_k(t)\}_{t\in\mathbb{R}_{\geq 0}}$ is a martingale for any $\theta \in \mathbb{R}$;

(ii) the error rate $\mathtt{ER}_k$ satisfies

$$\frac{1}{\exp(\frac{2\beta n}{\sigma^2}\eta_k) + 1} \leq \mathtt{ER}_k \leq \frac{\exp(\frac{2\beta n}{\sigma^2}(\eta_k + \frac{\beta n}{\sigma^2 \mu_k})) - 1}{\exp(\frac{4\beta n}{\sigma^2}\eta_k) - 1};$$

(iii) the stochastic process $\{x_k(t) - \beta t\}_{t\in\mathbb{R}_{\geq 0}}$ is a martingale;

(iv) the decision time $\mathbb{E}[T_k]$ satisfies

$$\frac{\eta_k}{\beta}\frac{1 - 2\exp(\frac{2\beta n}{\sigma^2}(\eta_k + \frac{\beta n}{\sigma^2\mu_k})) + \exp(\frac{4\beta n}{\sigma^2}\eta_k)}{\exp(\frac{4\beta n}{\sigma^2}\eta_k) - 1} \leq \mathbb{E}[T_k] \leq \frac{\eta_k}{\beta}\frac{\exp(\frac{2\beta n}{\sigma^2}\eta_k) - 1}{\exp(\frac{2\beta n}{\sigma^2}\eta_k) + 1}.$$

Proof: We start by establishing statement (i). In order to prove that the stochastic process $\{y_k(t)\}_{t\in\mathbb{R}_{\geq 0}}$ is a martingale, we need to show that for each $t \in \mathbb{R}_{\geq 0}$, and for some $s \leq t$: (a) $y_k(t)$ is measurable with respect to $\mathcal{F}_t^k$, (b) $\mathbb{E}[|y_k(t)|] < +\infty$, and (c) $\mathbb{E}[y_k(t)|\mathcal{F}_s^k] = y_k(s)$. The measurability condition (a) can be easily verified. To establish condition (b), it suffices to show that $\mathbb{E}[e^{\theta x_k(t)}] < +\infty$. From equation (3), $x_k(t)$ is normally distributed with mean βt and variance $\varsigma_k^2(t)$. Moreover, $\mathbb{E}[e^{\theta x_k(t)}]$ is the associated moment generating function. Consequently, $\mathbb{E}[e^{\theta x_k(t)}] = \exp(\beta t\theta + \varsigma_k^2(t)\theta^2/2) < +\infty$. To establish condition (c), we observe from the effective DDM (4) that $x_k(t)|\mathcal{F}_s^k$ is a normally distributed random variable with mean $x_k(s) + \beta(t-s)$ and variance $\sigma^2[\frac{t-s}{n} + \sum_{p=2}^{n} \frac{e^{-2\lambda_p s} - e^{-2\lambda_p t}}{2\lambda_p} {u_k^{(p)}}^2]$. Therefore,

$$\mathbb{E}[\exp(\theta x_k(t))|\mathcal{F}_s^k] = \exp\Big(\theta x_k(s) + \beta\theta(t-s) + \frac{1}{2}\theta^2\sigma^2\Big(\frac{t-s}{n} + \sum_{p=2}^{n} \frac{e^{-2\lambda_p s} - e^{-2\lambda_p t}}{2\lambda_p} {u_k^{(p)}}^2\Big)\Big). \tag{5}$$

It follows from equation (5) that

$$\mathbb{E}\Big[\exp\Big(\theta x_k(t) - \beta\theta t - \frac{1}{2}\theta^2\sigma^2\Big(\frac{t}{n} - \sum_{p=2}^{n} \frac{e^{-2\lambda_p t}}{2\lambda_p} {u_k^{(p)}}^2\Big)\Big)|\mathcal{F}_s^k\Big] = \exp\Big(\theta x_k(s) - \beta\theta s - \frac{1}{2}\theta^2\sigma^2\Big(\frac{s}{n} - \sum_{p=2}^{n} \frac{e^{-2\lambda_p s}}{2\lambda_p} {u_k^{(p)}}^2\Big)\Big),$$

which establishes condition (c).

We now establish statement (ii). We pick $\theta = -2\beta n/\sigma^2$, and consequently,

$$\bar{y}_k(t) = \exp\Big(-\frac{2\beta n}{\sigma^2}x_k(t) + \frac{2\beta^2 n^2}{\sigma^2} \sum_{p=2}^{n} \frac{e^{-2\lambda_p t}}{2\lambda_p} {u_k^{(p)}}^2\Big)$$

is a martingale. For the decision time T_k, $x_k(T_k) \in \{-\eta_k, +\eta_k\}$. Therefore,

$$\mathbb{E}[\bar{y}_k(T_k)] = ((1-\text{ER}_k)e^{-\frac{2\beta n}{\sigma^2}\eta_k} + \text{ER}_k e^{\frac{2\beta n}{\sigma^2}\eta_k}) \times e^{\frac{2\beta^2 n^2}{\sigma^2}\sum_{p=2}^n \frac{e^{-2\lambda_p T_k}}{2\lambda_p} u_k^{(p)2}}.$$

It follows from the optional stopping theorem [23] that $\mathbb{E}[\bar{y}_k(T_k)] = \bar{y}_k(0)$. Moreover, $0 \leq e^{-2\lambda_p T_k} \leq 1$, and consequently,

$$(1-\text{ER}_k)e^{-\frac{2\beta n}{\sigma^2}\eta_k} + \text{ER}_k e^{\frac{2\beta n}{\sigma^2}\eta_k} \leq e^{\frac{2\beta^2 n^2}{\sigma^4 \mu_k}} \leq ((1-\text{ER}_k)e^{-\frac{2\beta n}{\sigma^2}\eta_k} + \text{ER}_k e^{\frac{2\beta n}{\sigma^2}\eta_k})e^{\frac{2\beta^2 n^2}{\sigma^4 \mu_k}}. \quad (6)$$

Simplifying the inequalities (6) yields the desired bounds for the error rate.

To prove statement (iii), we observe that $x_k(t) - \beta t$ is measurable with respect to $\mathcal{F}_t$. It follows from Jensen's inequality that $\mathbb{E}[|x_k(t) - \beta t|] \leq \sqrt{\mathbb{E}[(x_k(t)-\beta t)^2]} = \varsigma_k(t) < +\infty$. Moreover, for any $s \leq t$, $\mathbb{E}[x_k(t) - \beta t | \mathcal{F}_s] = x_k(s) + \beta(t-s) - \beta t = x_k(s) - \beta s$. Hence, $\{x_k(t) - \beta t\}_{t \in \mathbb{R}_{\geq 0}}$ is a martingale.

We apply the optional stopping theorem [23] to the martingale $\{x_k(t) - \beta t\}_{t\in\mathbb{R}_{\geq 0}}$ to obtain $\mathbb{E}[x_k(T_k) - \beta T_k] = 0$. Hence, $\mathbb{E}[T_k] = \mathbb{E}[x_k(T_k)]/\beta = (1 - 2\text{ER}_k)\eta_k/\beta$. Substituting the lower and the upper bound for ER_k yields the upper and the lower bound for the expected decision time, respectively. ■

*Remark 2 (**Comparison with centralized decision maker**):* For a centralized decision maker, the effective drift and diffusion rates are β and σ^2/n, respectively. Thus, the lower bound on the error rate and the upper bound on the expected decision time in Theorem 1 correspond to a centralized decision maker. □

C. First passage time distribution

We now determine the first passage time distributions for the effective DDM (4). The first passage times T_k^+ and T_k^- associated with the thresholds $+\eta_k$ and $-\eta_k$, respectively, are defined by

$$T_k^+ = \inf\{t \in \mathbb{R}_{\geq 0} \mid x_k(t) \geq +\eta_k\}, \text{ and}$$
$$T_k^- = \inf\{t \in \mathbb{R}_{\geq 0} \mid x_k(t) \leq -\eta_k\}.$$

We denote the probability measure associated with the effective DDM (4) with drift rate β by $\mathbb{P}_\beta$. Let $\Phi(\cdot)$ represent the cumulative distribution function of the standard normal random variable. We now state the following theorem about the first passage time distributions.

*Theorem 2 (**First passage times**):* For the effective DDM (4) and the first passage times T_k^+ and T_k^-, the following statements hold:

(i) the first passage time densities under the effective DDM with drift rate β and $-\beta$ satisfy
$$e^{\frac{2\beta\eta_k}{\sigma^2}} \leq \frac{\mathbb{P}_\beta(T_k^+ \in ds)}{\mathbb{P}_{-\beta}(T_k^+ \in ds)} \leq e^{\frac{2\beta\eta_k n}{\sigma^2}};$$

(ii) the first passage time distributions satisfy
$$1 - \Phi\Big(\frac{\eta_k - \beta t}{\varsigma_k(t)}\Big) + e^{\frac{2\beta\eta_k}{\sigma^2}}\Phi\Big(\frac{-\eta_k - \beta t}{\varsigma_k(t)}\Big) \leq \mathbb{P}_\beta(T_k^+ \leq t) \leq 1 - \Phi\Big(\frac{\eta_k - \beta t}{\varsigma_k(t)}\Big) + e^{\frac{2\beta\eta_k n}{\sigma^2}}\Phi\Big(\frac{-\eta_k - \beta t}{\varsigma_k(t)}\Big), \text{ and}$$
$$\Phi\Big(\frac{-\eta_k - \beta t}{\varsigma_k(t)}\Big) + e^{\frac{2\beta\eta_k}{\sigma^2}}\Big(1 - \Phi\Big(\frac{\eta_k - \beta t}{\varsigma_k(t)}\Big)\Big) \leq \mathbb{P}_\beta(T_k^- \leq t) \leq \Phi\Big(\frac{-\eta_k - \beta t}{\varsigma_k(t)}\Big) + e^{\frac{2\beta\eta_k n}{\sigma^2}}\Big(1 - \Phi\Big(\frac{\eta_k - \beta t}{\varsigma_k(t)}\Big)\Big).$$

Proof: We first establish statement (i). We note that

$$\mathbb{P}_\beta(T_k^+ \in ds) = \mathbb{E}_\beta[\mathbf{1}(T_k^+ \in ds)] = \mathbb{E}_{-\beta}[\texttt{lr}\, \mathbf{1}(T_k^+ \in ds)], \quad (7)$$

where $\mathbb{E}_{\pm\beta}$ represents the expected value under the effective DDM (4) with drift rate $\pm\beta$, $\mathbf{1}(\cdot)$ represents the indicator function, and $\texttt{lr}$ is the likelihood ratio of $x_k(t), t \in [0, s]$ under the effective DDM with drift rate $+\beta$ and $-\beta$, respectively. We now evaluate $\texttt{lr}$. We discretize the interval $[0, s]$ to obtain the increasing sequence $\{s_1, \ldots, s_m\}$, where $s_1 = 0$ and $s_m = s$. Let $\texttt{lr}_m$ be defined by

$$\texttt{lr}_m = \frac{\mathbb{P}_\beta(x_k(t_1) \in dx^1, \ldots, x_k(t_m) \in dx^m)}{\mathbb{P}_{-\beta}(x_k(t_1) \in dx^1, \ldots, x_k(t_m) \in dx^m)}.$$

We note that $\lim_{m\to+\infty} \texttt{lr}_m = \texttt{lr}$. It follows that $\texttt{lr}_m = \exp\big(\sum_{i=1}^m \frac{2\beta(x_k(t_{i+1}) - x_k(t_i))}{\sigma_k(t_i)^2}\big)$. Since, $0 \leq e^{-\lambda_p t_i} \leq 1$, it follows that

$$\frac{\sigma^2}{n} \leq \sigma_k(t_i)^2 \leq \sigma_k(0)^2 = \sigma^2, \text{ and}$$
$$\exp\Big(\frac{2\beta\eta_k}{\sigma^2}\Big) \leq \texttt{lr}_m \leq \exp\Big(\frac{2\beta\eta_k n}{\sigma^2}\Big).$$

We note that the bounds on $\texttt{lr}_m$ are independent of m and hence, hold for $\texttt{lr}$ as well. The bounds on $\texttt{lr}$ along with equation (7) establish statement (i).

To establish statement (ii), we note that

$$\begin{aligned}\mathbb{P}_\beta(T_k^+ \leq t) &= \mathbb{P}_\beta(T_k^+ \leq t, x_k(t) \geq \eta_k) + \mathbb{P}_\beta(T_k^+ \leq t, x_k(t) < \eta_k) \\ &= \mathbb{P}_\beta(x_k(t) \geq \eta_k) + \mathbb{P}_\beta(T_k^+ \leq t, x_k(t) < \eta_k). \quad (8)\end{aligned}$$

We now evaluate $\mathbb{P}_\beta(T_k^+ \leq t, x_k(t) < \eta_k)$. It follows from the definition of joint probability that

$$\begin{aligned}\mathbb{P}_\beta(T_k^+ \leq t, x_k(t) < \eta_k) &= \int_{s=0}^t \mathbb{P}_\beta(x_k(t) \leq \eta_k | x_k(s) = \eta_k)\mathbb{P}_\beta(T_k^+ \in ds) \\ &= \int_{s=0}^t \mathbb{P}_\beta(z^+(t-s) \leq 0)\mathbb{P}_\beta(T_k^+ \in ds) \\ &= \int_{s=0}^t \mathbb{P}_{-\beta}(z^-(t-s) \geq 0)\mathbb{P}_\beta(T_k^+ \in ds), \quad (9)\end{aligned}$$

where $z^\pm(t-s)$ is a normally distributed random variable with mean $\pm\beta(t-s)$ and variance $\sigma^2\big(\frac{t-s}{n} + \sum_{p=2}^n \frac{e^{-2\lambda_p s} - e^{-2\lambda_p t}}{2\lambda_p} u_k^{(p)2}\big)$. It follows from the first statement and equation (9) that

$$\begin{aligned}e^{\frac{2\beta\eta_k}{\sigma^2}} \int_{s=0}^t \mathbb{P}_{-\beta}(z^-(t-s) \geq 0)\mathbb{P}_{-\beta}(T_k^+ \in ds) &\leq \mathbb{P}_\beta(T_k^+ \leq t, x_k(t) < \eta_k) \\ &\leq e^{\frac{2\beta\eta_k n}{\sigma^2}} \int_{s=0}^t \mathbb{P}_{-\beta}(z^-(t-s) \geq 0)\mathbb{P}_{-\beta}(T_k^+ \in ds).\end{aligned}$$

Consequently,

$$\begin{aligned}e^{\frac{2\beta\eta_k}{\sigma^2}}\mathbb{P}_{-\beta}(T_k^+ \leq t, x_k(t) \geq \eta_k) &\leq \mathbb{P}_\beta(T_k^+ \leq t, x_k(t) \geq \eta_k) \\ &\leq e^{\frac{2\beta\eta_k n}{\sigma^2}}\mathbb{P}_{-\beta}(T_k^+ \leq t, x_k(t) \geq \eta_k).\end{aligned}$$

Furthermore, $\mathbb{P}_{-\beta}(T_k^+ \leq t, x_k(t) > \eta_k) = \mathbb{P}_{-\beta}(x_k(t) \geq \eta_k) = \mathbb{P}_{\beta}(x_k(t) \leq -\eta_k)$. Hence,

$$e^{\frac{2\beta\eta_k}{\sigma^2}} \leq \frac{\mathbb{P}_{-\beta}(T_k^+ \leq t, x_k(t) > \eta_k)}{\mathbb{P}_{\beta}(x_k(t) \leq -\eta_k)} \leq e^{\frac{2\beta\eta_k n}{\sigma^2}}.$$

Consequently, from equation (8), we have the desired bounds on $\mathbb{P}_{\beta}(T_k^+ \leq t)$. Bounds on $\mathbb{P}_{\beta}(T_k^- \leq t)$ can be established similarly. ■

*Corollary 3 (**Asymptotics**):* For the effective DDM (4) and the free response paradigm, the following statements hold in the limit $\eta_k \to +\infty$

(i) the decision time $T_k \to +\infty$ almost surely;

(ii) the upper bound on error rate and the lower bound on expected decision time in Theorem 1 are achieved.

Proof: We start by establishing statement (i). We note that $\mathbb{P}(T_k \leq t) \leq \mathbb{P}(T_k^+ \leq t) + \mathbb{P}(T_k^- \leq t)$. It follows from Theorem 2 that $\mathbb{P}(T_k^+ \leq t) + \mathbb{P}(T_k^- \leq t) \to 0^+$ as $\eta_k \to +\infty$. Consequently, $\mathbb{P}(T_k \leq t) \to 0^+$ as $\eta_k \to +\infty$, i.e., $T_k \to +\infty$ in probability as $\eta_k \to +\infty$. Therefore, there exists an increasing subsequence of η_k for which $T_k \to +\infty$ almost surely. Moreover, T_k is a non-decreasing function of η_k, hence $T_k \to +\infty$ almost surely as $\eta_k \to +\infty$.

The second statement follows by observing that as $\eta_k \to +\infty$, $e^{-\lambda_p T_k} \to 0^+$ almost surely, and hence, the upper bound on error rate and the lower bound on expected decision time are achieved. ■

D. Optimal threshold design

In this section, we elucidate on various threshold selection mechanisms for individuals in the group. We first discuss the Wald-like threshold selection mechanism that is suited for threshold selection in engineering applications. Then, we discuss Bayes risk minimizing and reward rate maximizing mechanisms that are plausible threshold selection methods in human decision making.

Wald-like mechanism: In the classical sequential hypothesis testing problem [24], the thresholds are designed such that the probability of error is below a desired value. In a similar spirit, we can pick threshold η_k such that the probability of error is below a desired value $\alpha_k \in (0,1)$. It follows from bounds on the error rate in Theorem 1 that such a threshold η_k is the solution of the following transcendental equation:

$$e^{\frac{2\beta^2 n^2}{\sigma^4 \mu_k}} - (1-\alpha_k) e^{\frac{-2\beta n}{\sigma^2}\eta_k} - \alpha_k e^{\frac{2\beta n}{\sigma^2}\eta_k} = 0. \qquad (10)$$

It follows that as $\alpha_k \to 0^+$, equation (10) holds only if $\eta_k \to +\infty$. Under such asymptotic regime $e^{\frac{-2\beta n}{\sigma^2}\eta_k} \to 0^+$ and the desired threshold is approximately equal to $\eta_k^{\text{wald}} \approx \frac{\beta n}{2\sigma^2 \mu_k} - \frac{\sigma^2}{2\beta n}\log \alpha_k$.

Bayes risk minimizing mechanism: The Bayes risk minimization is one of the plausible mechanisms for threshold selection for humans [5]. In this mechanism, the threshold η_k is selected to minimize the Bayes risk ($\mathtt{BR}_k$) defined by

$$\mathtt{BR}_k = c_k \mathtt{ER}_k + d_k \mathbb{E}[T_k],$$

where $c_k, d_k \in \mathbb{R}_{\geq 0}$ are parameters that are determined from empirical data [5]. Using the asymptotic expressions in Theorem 1, we have

$$\mathtt{BR}_k = \Big(c_k - \frac{2d_k\eta_k}{\beta}\Big)\Big(\frac{\exp(\frac{2\beta n}{\sigma^2}(\eta_k + \frac{\beta n}{\sigma^2 \mu_k})) - 1}{\exp(\frac{4\beta n}{\sigma^2}\eta_k) - 1}\Big) + \frac{d_k \eta_k}{\beta}.$$

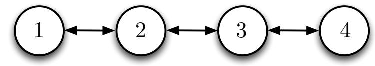

Fig. 1. Connection topology of individuals in the numerical example.

$\mathtt{BR}_k$ is a univariate function of the threshold η_k and can be numerically minimized to determine an optimal threshold.

Reward rate maximizing mechanism: Another plausible mechanism for threshold selection in humans is reward rate maximization [5]. The reward rate ($\mathtt{RR}_k$) is defined by

$$\mathtt{RR}_k = \frac{1 - \mathtt{ER}_k}{\mathbb{E}[T_k] + T_{\text{motor}} + D + \mathtt{ER}_k D_p},$$

where T_{motor} is the motor time associated with decision making process, D is the response time, and D_p is the additional time that the human takes after an erroneous decision (see [5] for detailed description of the parameters). Thus, the reward rate for the k-th individual is

$$\mathtt{RR}_k = \frac{1 - \mathtt{ER}_k}{(1 - 2\mathtt{ER}_k)\frac{\eta_k}{\beta} + T_{\text{motor}} + D + \mathtt{ER}_k D_p},$$

where $\mathtt{ER}_k$ is the asymptotic expression for the error rate for k-th individual. Similar to the Bayesian risk, the reward rate is also a univariate function of the threshold η_k and can be numerically maximized to determine an optimal threshold.

IV. Numerical Examples

We consider a set of four individuals with interaction topology in Figure 1. The drift and the diffusion rate for each individual are 0.2 and unity, respectively. Error rates and reaction times for individual 1 and 2 are shown in Figures 2 and 3, respectively. Note that the upper bound on the error rate and the lower bound on the expected decision time better predict the associated quantities for the coupled DDM as compared to the asymptotic predictors that correspond to a centralized decision maker.

First passage time distributions associated with unity threshold are shown in Figure 4. The upper bound on the distribution function for the effective DDM is very close to the distribution function obtained from Monte Carlo simulations, while the distribution function associated with a centralized decision maker differs significantly from it. The primary reason for this difference is that the noise in the centralized case is very low. In particular, the diffusion rate for a centralized decision maker is σ^2/n and with increasing n, the distribution function converges to a step function, while for the coupled DDM the diffusion rate at each node reaches σ^2/n only asymptotically.

V. Conclusions and Future Directions

In this paper, we studied the speed-accuracy trade-off in collective decision making in human groups using the context of two alternative choice tasks. We focused on the free response decision making paradigm in which each individual takes their time to make a decision. We derived approximate bounds on error rates, expected decision times, and first passage time distributions for each individual in the network. We also discussed various threshold selection criteria.

There are several possible extensions to this work. First, the mean-field type approximations considered in this paper

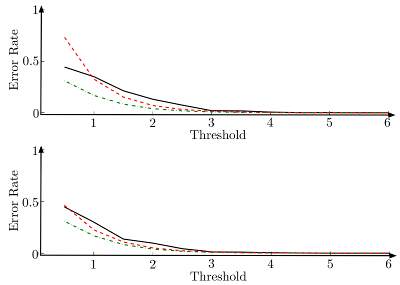

Fig. 2. Error rates. The top and the bottom figures show error rates for individuals 1 and 2, respectively. The solid black lines represent error rates for the coupled DDM obtained using Monte Carlo simulations, the dashed red lines and green dashed-dotted lines represent the upper and lower bound obtained for the effective DDM.

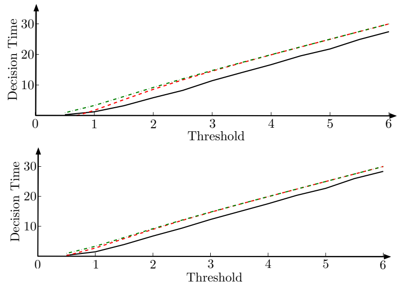

Fig. 3. Expected decision times. The top and the bottom figures show expected decision times for individuals 1 and 2, respectively. The solid black lines represent decision times for the coupled DDM obtained using Monte Carlo simulations, the dashed red lines and green dashed-dotted lines represent the lower and upper bound obtained for the effective DDM.

determine an effective DDM by matching the evidence distribution for the effective DDM with the marginal distribution of an individual's evidence. It is of interest to explore other possible mean-field type approximations, for instance, the effective DDM can be selected such that it is closest to the original coupled distribution in the sense of Kullback-Leibler divergence. Second, in several decision making scenarios, stochastic models close to DDM, e.g., Ornstein-Uhlenbeck process, capture the information aggregation process. It is of interest to extend the analysis in this paper to such models. Third, in this paper, we considered two alternative choice tasks. Diffusion models for multiple alternative choice tasks are available [25], and it is of interest to extend this work to multiple alternative choice tasks. Fourth, in the spirit of the centrality measures [13] in the interrogation paradigm, it is of interest to explore the notion of centrality in the free response paradigm.

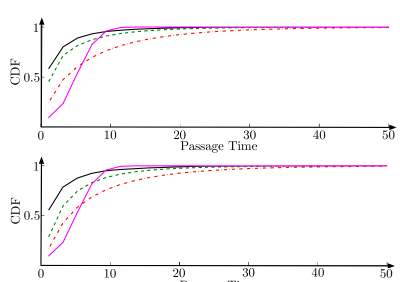

Fig. 4. Passage time distributions. The top and the bottom figures show passage time distributions for individuals 1 and 2, respectively. The solid black lines represent passage time distribution for the coupled DDM obtained using Monte Carlo simulations, the dashed-dotted red lines and green dashed lines represent the lower and upper bound obtained for the effective DDM, and the magenta line with dots represents passage time distribution for a centralized decision maker.

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