Randomized Sensor Selection in Sequential Hypothesis Testing

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***Abstract*—We consider the problem of sensor selection for time-optimal detection of a hypothesis. We consider a group of sensors transmitting their observations to a fusion center. The fusion center considers the output of only one randomly chosen sensor at the time, and performs a sequential hypothesis test. We study a sequential multiple hypothesis test with randomized sensor selection strategy. We incorporate the random processing times of the sensors to determine the asymptotic performance characteristics of this test. For three distinct performance metrics, we show that, for a generic set of sensors and binary hypothesis, the time-optimal policy requires the fusion center to consider at most two sensors. We also show that for the case of multiple hypothesis, the time-optimal policy needs at most as many sensors to be observed as the number of underlying hypotheses.**

***Index Terms*—Sensor selection, decision making, SPRT, MSPRT, sequential hypothesis testing, linear-fractional programming.**

I. Introduction

In today's information-rich world, different sources are best informers about different topics. If the topic under consideration is well known beforehand, then one chooses the best source. Otherwise, it is not obvious what source or how many sources one should observe. This need to identify sensors (information sources) to be observed in decision making problems is found in many common situations, e.g., when deciding which news channel to follow. When a person decides what information source to follow, she relies in general upon her experience, i.e., one knows through experience what combination of news channels to follow.

In engineering applications, a reliable decision on the underlying hypothesis is made through repeated measurements. Given infinitely many observations, decision making can be performed accurately. Given a cost associated to each observation, a well-known trade-off arises between accuracy and number of iterations. Various sequential hypothesis tests have been proposed to detect the underlying hypothesis within a given degree of accuracy. There exist two different classes of sequential tests. The first class includes sequential tests developed from the dynamic programming point of view. These tests are optimal and, in general, difficult to implement [5]. The second class consists of easily-implementable and asymptotically-optimal sequential tests; a widely-studied example is the Sequential Probability Ratio Test (SPRT) for binary hypothesis testing and its extension, the Multi-hypothesis Sequential Probability Ratio Test (MSPRT).

In this paper, we consider the problem of quickest decision making and sequential probability ratio tests. Recent advances in cognitive psychology [7] show that human performance in decision making tasks, such as the "two-alternative forced choice task," is well modeled by a drift diffusion process, i.e., by the continuous-time version of SPRT. Roughly speaking, modeling decision making as an SPRT process may be appropriate even for situations in which a human is making the decision.

Sequential hypothesis testing and quickest detection problems have been vastly studied [18], [4]. The SPRT for binary decision making was introduced by Wald in [22], and was extended by Armitage to multiple hypothesis testing in [1]. The Armitage test, unlike the SPRT, is not necessarily optimal [5]. Various other tests for multiple hypothesis testing have been developed throughout the years; see [20] and references there in. A sequential test for multiple hypothesis testing was developed in [5], and [11], which provides with an asymptotic expression for the expected sample size. This sequential test is called the MSPRT and reduces to the SPRT in case of binary hypothesis. We consider MSPRT for multiple hypothesis testing in this paper.

Recent years have witnessed a significant interest in the problem of sensor selection for optimal detection and estimation. Tay et al [21] discuss the problem of censoring sensors for decentralized binary detection. They assess the quality of sensor data by the Neyman-Pearson and a Bayesian binary hypothesis test and decide on which sensors should transmit their observation at that time instant. Gupta et al [13] focus on stochastic sensor selection and minimize the error covariance of a process estimation problem. Isler et al [15] propose geometric sensor selection schemes for error minimization in target detection. Debouk et al [10] formulate a Markovian decision problem to ascertain some property in a dynamical system, and choose sensors to minimize the associated cost. Williams et at [24] use an approximate dynamic program over a rolling time horizon to pick a sensor-set that optimizes the information-communication trade-off. Wang et al [23] design entropy-based sensor selection algorithms for target localization. Joshi et al [16] present a convex optimization-based heuristic to select multiple sensors for optimal parameter estimation. Bajović et al [3] discuss sensor selection problems for Neyman-Pearson binary hypothesis testing in wireless sensor networks. Castañón [9] study an iterative search problem as a hypothesis testing problem over a fixed horizon.

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A third and last set of references related to this paper are those on linear-fractional programming. Various iterative and cumbersome algorithms have been proposed to optimize linear-fractional functions [8], [2]. In particular, for the problem of minimizing the sum and the maximum of linear-fractional functionals, some efficient iterative algorithms have been proposed, including the algorithms by Falk et al [12] and by Benson [6].

In this paper, we analyze the problem of time-optimal sequential decision making in the presence of multiple switching sensors and determine a randomized sensor selection strategy to achieve the same. We consider a sensor network where all sensors are connected to a fusion center. Such topology is found in numerous sensor networks with cameras, sonars or radars, where the fusion center can communicate with any of the sensors at each time instant. The fusion center, at each instant, receives information from only one sensor. Such a situation arises when we have interfering sensors (e.g., sonar sensors), a fusion center with limited attention or information processing capabilities, or sensors with shared communication resources. The sensors may be heterogeneous (e.g., a camera sensor, a sonar sensor, a radar sensor, etc), hence, the time needed to collect, transmit, and process data may differ significantly for these sensors. The fusion center implements a sequential hypothesis test with the gathered information. We consider the MSPRT for multiple hypothesis testing. First, we develop a version of the MSPRT algorithm in which the sensor is randomly switched at each iteration, and determine the expected time that this test requires to obtain a decision within a given degree of accuracy. Second, we identify the set of sensors that minimize the expected decision time. We consider three different cost functions, namely, the conditioned decision time, the worst case decision time, and the average decision time. We show that the expected decision time, conditioned on a given hypothesis, using these sequential tests is a linear-fractional function defined on the probability simplex. We exploit the special structure of our domain (probability simplex), and the fact that our data is positive to tackle the problem of the sum and the maximum of linear-fractional functionals analytically. Our approach provides insights into the behavior of these functions. The major contributions of this paper are:

i) We develop a version of the MSPRT where the sensor is selected randomly at each observation.
ii) We determine the asymptotic expressions for the thresholds and the expected sample size for this sequential test.
iii) We incorporate the random processing time of the sensors into these models to determine the expected decision time.
iv) We show that, to minimize the conditioned expected decision time, the optimal policy requires only one sensor to be observed.
v) We show that, for a generic set of sensors and M underlying hypotheses, the optimal average decision time policy requires the fusion center to consider at most M sensors.
vi) For the binary hypothesis case, we identify the optimal set of sensors in the worst case and the average decision time minimization problems. Moreover, we determine an optimal probability distribution for the sensor selection.
vii) In the worst case and the average decision time minimization problems, we encounter the problem of minimization of sum and maximum of linear-fractional functionals. We treat these problems analytically, and provide insight into their optimal solutions.

The remainder of the paper is organized in following way. Some preliminaries are presented in Section II. In Section III, we present the problem setup. We develop the randomized sensor selection version of the MSPRT procedure in Section IV. In Section V, we formulate the optimization problems for time-optimal sensor selection, and determine their solution. We elucidate the results obtained through numerical examples in Section VI. Our concluding remarks are in Section VII.

II. Preliminaries

A. Linear-fractional function

Given parameters $A \in \mathbb{R}^{l\times p}$, $B \in \mathbb{R}^l$, $c \in \mathbb{R}^p$, and $d \in \mathbb{R}$, the function $g : \{z \in \mathbb{R}^p \mid c^T z + d > 0\} \to \mathbb{R}^l$, defined by

$$g(x) = \frac{Ax + B}{c^T x + d},$$

is called a *linear-fractional function* [8]. A linear-fractional function is quasi-convex as well as quasi-concave. In particular, if $l = 1$, then any scalar linear-fractional function g satisfies

$$\begin{aligned} g(\nu x + (1-\nu)y) &\le \max\{g(x), g(y)\}, \\ g(\nu x + (1-\nu)y) &\ge \min\{g(x), g(y)\}, \end{aligned} \tag{1}$$

for all $\nu \in [0,1]$ and $x, y \in \{z \in \mathbb{R}^p \mid c^T z + d > 0\}$.

B. Kullback-Leibler divergence

Given two probability mass functions $f_1 : \mathcal{S} \to \mathbb{R}_{\ge 0}$ and $f_2 : \mathcal{S} \to \mathbb{R}_{\ge 0}$, where $\mathcal{S}$ is some countable set, the Kullback-Leibler divergence $\mathcal{D} : \mathcal{L}^1 \times \mathcal{L}^1 \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{D}(f_1, f_2) = \mathbb{E}_{f_1}\left[\log \frac{f_1(X)}{f_2(X)}\right] = \sum_{x \in \operatorname{supp}(f_1)} f_1(x) \log \frac{f_1(x)}{f_2(x)},$$

where $\mathcal{L}^1$ is the set of integrable functions and $\operatorname{supp}(f_1)$ is the support of f_1. It is known that $0 \le \mathcal{D}(f_1, f_2) \le +\infty$, that the lower bound is achieved if and only if $f_1 = f_2$, and that the upper bound is achieved if and only if the support of f_2 is a strict subset of the support of f_1. Note that equivalent statements can be given for probability density functions.

C. Multi-hypothesis Sequential Probability Ratio Test

The MSPRT for multiple hypothesis testing was introduced in [5], [11]. It is described as follows. Given M hypotheses with probability density functions $f^k(y) := f(y|H_k)$, $k \in \{0, \dots, M-1\}$, the posterior probability after τ observations y_t, $t \in \{1, \dots, \tau\}$ is given by

$$p^k_\tau = \mathbb{P}(H_k | y_1, \dots, y_\tau) = \frac{\prod_{t=1}^{\tau} f^k(y_t)}{\sum_{j=0}^{M-1} \prod_{t=1}^{\tau} f^j(y_t)}. \tag{2}$$

Because the denominator is same for each k, the hypothesis with maximum posterior probability p_τ^k at any time τ is the one maximizing the numerator $\prod_{t=1}^{\tau} f^j(y_t)$. Given these observations, the MSPRT is described in Algorithm 1. The

Algorithm 1 Multi-hypothesis sequential probability ratio test

1: at time $\tau \in \mathbb{N}$, collect sample y_τ
2: compute the posteriors p_τ^k, $k \in \{0, \dots, M-1\}$ as in (2)
% decide only if a threshold is crossed
3: **if** $p_\tau^h > \dfrac{1}{1+\eta_h}$ for at least one $h \in \{0, \dots, M-1\}$,
4: **then** accept H_k with maximum p_τ^k satisfying step `3:`,
5: **else** continue sampling (step `1:`)

thresholds η_k are designed as functions of the frequentist error probabilities (i.e., the probabilities to accept a given hypothesis wrongly) α_k, $k \in \{0, \dots, M-1\}$. Specifically, the thresholds are given by

$$\eta_k = \frac{\alpha_k}{\gamma_k}, \tag{3}$$

where $\gamma_k \in]0,1[$ is a constant function of f^k (see [5]), and $]\cdot,\cdot[$ represents the open interval.

Let $\eta_{\max} = \max\{\eta_j \mid j \in \{0, \dots, M-1\}\}$. It is known [5] that the expected sample size of the MSPRT N_d, conditioned on a hypothesis, satisfies

$$\mathbb{E}\left[N_d | H_k\right] \to \frac{-\log \eta_k}{\mathcal{D}^*(k)}, \quad \text{as } \eta_{\max} \to 0^+,$$

where $\mathcal{D}^*(k) = \min\{\mathcal{D}(f^k, f^j) \mid j \in \{0, \dots, M-1\}, j \neq k\}$ is the minimum Kullback-Leibler divergence from the distribution f^k to all other distributions f^j, $j \neq k$.

The MSPRT is an easily-implementable hypothesis test and is shown to be asymptotically optimal in [5], [11]. For $M = 2$, the MSPRT reduces to SPRT which is optimal in the sense that it minimizes the expected sample size required to decide within a given error probability.

III. Problem setup

We consider a group of n agents (e.g., robots, sensors, or cameras), which take measurements and transmit them to a fusion center. We generically call these agents "sensors." We identify the fusion center with a person supervising the agents, and call it the "supervisor." The goal of the supervisor is to decide, based on the measurements it receives, which one of M alternative hypotheses or "states of nature" is correct. To do so, the supervisor implements the MSPRT with the collected observations. Given pre-specified accuracy thresholds, the supervisor aims to make a decision in minimum time.

We assume that there are more sensors than hypotheses (i.e., $n > M$), and that only one sensor can transmit to the supervisor at each (discrete) time instant. Equivalently, the supervisor can process data from only one of the n sensors at each time. Thus, at each time, the supervisor must decide which sensor should transmit its measurement. This setup also models a sequential search problem, where one out of n sensors is sequentially activated to establish the most

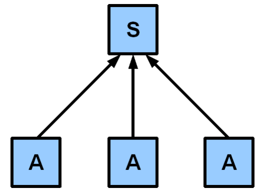

Fig. 1. The agents A transmit their observation to the supervisor S, one at the time. The supervisor performs a sequential hypothesis test to decide on the underlying hypothesis.

likely intruder location out of M possibilities; see [9] for a related problem. In this paper, our objective is to determine the optimal sensor(s) that the supervisor must observe in order to minimize the decision time.

We adopt the following notation. Let $\{H_0, \dots, H_{M-1}\}$ denote the $M \geq 2$ hypotheses. The time required by sensor $s \in \{1, \dots, n\}$ to collect, process and transmit its measurement is a random variable $T_s \in \mathbb{R}_{>0}$, with finite first and second moment. We denote the mean processing time of sensor s by $\bar{T}_s \in \mathbb{R}_{>0}$. Let $s_t \in \{1, \dots, n\}$ indicate which sensor transmits its measurement at time $t \in \mathbb{N}$. The measurement of sensor s at time t is $y(t,s)$. For the sake of convenience, we denote $y(t, s_t)$ by y_t. For $k \in \{0, \dots, M-1\}$, let $f_s^k : \mathbb{R} \to \mathbb{R}$ denote the probability density function of the measurement y at sensor s conditioned on the hypothesis H_k. Let $f^k : \{1, \dots, n\} \times \mathbb{R} \to \mathbb{R}$ be the probability density function of the pair (s, y), conditioned on hypothesis H_k. For $k \in \{0, \dots, M-1\}$, let α_k denote the desired bound on probability of incorrect decision conditioned on hypothesis H_k. We make the following standard assumption:

Conditionally-independent observations: Conditioned on hypothesis H_k, the measurement $y(t,s)$ is independent of $y(\bar{t}, \bar{s})$, for $(t,s) \neq (\bar{t}, \bar{s})$.

We adopt a *randomized strategy* in which the supervisor chooses a sensor randomly at each time instant; the probability to choose sensor s is stationary and given by q_s, for $s \in \{1, \dots, n\}$. Also, the supervisor uses the data collected from the randomized sensors to execute a multi-hypothesis sequential hypothesis test. For the stationary randomized strategy, note that $f^k(s,y) = q_s f_s^k(y)$. We study our proposed randomized strategy under the following assumptions about the sensors.

Distinct sensors: There are no two sensors with identical conditioned probability density $f_s^k(y)$ and mean processing time $\bar{T}_s$. (If there are such sensors, we club them together in a single node, and distribute the probability assigned to that node equally among them.)

Finitely-informative sensors: Each sensor $s \in \{1, \dots, n\}$ has the following property: for any two hypotheses $k, j \in \{0, \dots, M-1\}$, $k \neq j$,

i) the support of f_s^k is equal to the support of f_s^j,
ii) $f_s^k \neq f_s^j$ almost surely in f_s^k, and
iii) conditioned on hypothesis H_k, the first and second moment of $\log(f_s^k(Y)/f_s^j(Y))$ are finite.

Remark 1: The finitely-informative sensors assumption is equivalently restated as follows: each sensor $s \in \{1,\dots,n\}$ satisfies $0 < \mathcal{D}(f_s^k, f_s^j) < +\infty$ for any two hypotheses $k, j \in \{0,\dots,M-1\}$, $k \neq j$. $\square$

Remark 2: We study a stationary policy because it is simple to implement, it is amenable to rigorous analysis and it has intuitively-appealing properties (e.g., we show that the optimal stationary policy requires the observation of only as many sensors as the number of hypothesis). On the contrary, if we do not assume a stationary policy, the optimal solution would be based on dynamic programming and, correspondingly, would be complex to implement, analytically intractable, and would lead to only numerical results. $\square$

IV. MSPRT WITH RANDOMIZED SENSOR SELECTION

We call the MSPRT with the data collected from n sensors while observing only one sensor at a time as the MSPRT with randomized sensor selection. For each sensor s, define $\mathcal{D}_s^*(k) = \min\{\mathcal{D}(f_s^k, f_s^j) \mid j \in \{0,\dots,M-1\}, j \neq k\}$. The sensor to be observed at each time is determined through a randomized policy, and the probability of choosing sensor s is stationary and given by q_s. Assume that the sensor $s_t \in \{1,\dots,n\}$ is chosen at time instant t, then the posterior probability after the observations y_t, $t \in \{1,\dots,\tau\}$, is given by

$$p_\tau^k = \mathbb{P}(H_k|y_1,\dots,y_\tau) = \frac{\prod_{t=1}^{\tau} f^k(s_t, y_t)}{\sum_{j=0}^{M-1}\prod_{t=1}^{\tau} f^j(s_t, y_t)}$$
$$= \frac{\prod_{t=1}^{\tau} q_{s_t} f_{s_t}^k(y_t)}{\sum_{j=0}^{M-1}\prod_{t=1}^{\tau} q_{s_t} f_{s_t}^j(y_t)} = \frac{\prod_{t=1}^{\tau} f_{s_t}^k(y_t)}{\sum_{j=0}^{M-1}\prod_{t=1}^{\tau} f_{s_t}^j(y_t)}, \quad (4)$$

and, at any given time τ, the hypothesis with maximum posterior probability p_τ^k is the one maximizing $\prod_{t=1}^{\tau} f_{s_t}^k(y_t)$. Note that the sequence $\{(s_t, y_t)\}_{t\in\mathbb{N}}$ is an i.i.d. realization of the pair (s, Y_s), where Y_s is the measurement of sensor s.

For thresholds η_k, $k \in \{0,\dots,M-1\}$, defined in equation (3), the MSPRT with randomized sensor selection is defined identically to the Algorithm 1, where the first two instructions (steps `1:` and `2:`) are replaced by:

1: at time $\tau \in \mathbb{N}$, select a random sensor s_τ according to the probability vector q and collect a sample y_τ
2: compute the posteriors p_τ^k, $k \in \{0,\dots,M-1\}$ as in (4)

Lemma 1 (Asymptotics): Assume finitely informative sensors $\{1,\dots,n\}$. Conditioned on hypothesis H_k, $k \in \{0,\dots,M-1\}$, the sample size for decision $N_d \to \infty$ almost surely as $\eta_{\max} \to 0^+$.

Proof:

$$\mathbb{P}(N_d \le \tau|H_k)$$
$$= \mathbb{P}\Big(\min_{a\in\{1,\dots,\tau\}} \sum_{\substack{j=1\\ j\neq v}}^{M-1}\prod_{t=1}^{a}\frac{f_{s_t}^j(y_t)}{f_{s_t}^v(y_t)} < \eta_v, \text{ for some } v \in \{0,\dots,M-1\}\Big|H_k\Big)$$
$$\le \mathbb{P}\Big(\min_{a\in\{1,\dots,\tau\}} \prod_{t=1}^{a}\frac{f_{s_t}^j(y_t)}{f_{s_t}^v(y_t)} < \eta_v, \text{ for some } v, \text{ and any } j \neq v\Big|H_k\Big)$$
$$= \mathbb{P}\Big(\max_{a\in\{1,\dots,\tau\}} \sum_{t=1}^{a}\log\frac{f_{s_t}^v(y_t)}{f_{s_t}^j(y_t)} > -\log\eta_v, \text{ for some } v, \text{ and any } j \neq v\Big|H_k\Big)$$
$$\le \sum_{\substack{v=0\\ v\neq k}}^{M-1}\mathbb{P}\Big(\max_{a\in\{1,\dots,\tau\}} \sum_{t=1}^{a}\log\frac{f_{s_t}^v(y_t)}{f_{s_t}^k(y_t)} > -\log\eta_v\Big|H_k\Big) + \mathbb{P}\Big(\max_{a\in\{1,\dots,\tau\}} \sum_{t=1}^{a}\log\frac{f_{s_t}^k(y_t)}{f_{s_t}^{j^*}(y_t)} > -\log\eta_k\Big|H_k\Big),$$

for some $j^* \in \{0,\dots,M-1\}\setminus\{k\}$. Observe that since $0 < \mathcal{D}(f_s^k, f_s^j) < \infty$, for each $j, k \in \{0,\dots,M-1\}, j \neq k$, and $s \in \{1,\dots,n\}$, the above right hand side goes to zero as $\eta_{\max} \to 0^+$. Hence, conditioned on a hypothesis H_k, the sample size for decision $N_d \to \infty$ in probability. This means that there exists a subsequence such that $N_d \to \infty$ almost surely. We further observe that N_d is a non decreasing as we decrease $\eta_{\max}$. Hence, conditioned on hypothesis H_k, $N_d \to \infty$, almost surely, as $\eta_{\max} \to 0^+$. $\blacksquare$

Lemma 2 (Theorem 5.2, [5]): Assume the sequences of random variables $\{Z_t^j\}_{t\in\mathbb{N}}$, $j \in \{1,\dots,d\}$, converge to μ_j almost surely as $t \to \infty$, with $0 < \min_{j\in\{1,\dots,d\}} \mu_j < \infty$. Then as $t \to \infty$, almost surely,

$$-\frac{1}{t}\log\Big(\sum_{j=1}^{d} e^{-tZ_t^j}\Big) \to \min_{j\in\{1,\dots,d\}} \mu_j.$$

$\square$

Lemma 3 (Corollary 7.4.1, [19]): Let $\{Z_t\}_{t\in\mathbb{N}}$ be independent sequence of random variables satisfying $\mathbb{E}[Z_t^2] < \infty$, for all $t \in \mathbb{N}$, and $\{b_t\}_{t\in\mathbb{N}}$ be a monotone sequence such that $b_t \to \infty$ as $t \to \infty$. If $\sum_{i=1}^{\infty} \text{Var}\,(Z_i/b_i) < \infty$, then

$$\frac{\sum_{i=1}^{t} Z_i - \mathbb{E}[\sum_{i=1}^{t} Z_i]}{b_t} \to 0, \text{ almost surely as } t \to \infty.$$

$\square$

Lemma 4 (Theorem 2.1 in [14]): Let $\{Z_t\}_{t\in\mathbb{N}}$ be a sequence of random variables and $\{\tau(a)\}_{a\in\mathbb{R}_{\ge 0}}$ be a family of positive, integer valued random variables. Suppose that $Z_t \to Z$ almost surely as $t \to \infty$, and $\tau(a) \to \infty$ almost surely as $a \to \infty$. Then $Z_{\tau(a)} \to Z$ almost surely as $a \to \infty$. $\square$

We now present the main result of this section, whose proof is a variation of the proofs for MSPRT in [5].

Theorem 1 (MSPRT with randomized sensor selection): Assume finitely-informative sensors $\{1,\dots,n\}$, and independent observations conditioned on hypothesis H_k, $k \in \{0,\dots,M-1\}$. For the MSPRT with randomized sensor selection, the following statements hold:

i) Conditioned on a hypothesis, the sample size for decision N_d is finite almost surely.

ii) Conditioned on hypothesis H_k, the sample size for decision N_d, as $\eta_{\max} \to 0^+$, satisfies

$$\frac{N_d}{-\log \eta_k} \to \frac{1}{\sum_{s=1}^n q_s \mathcal{D}_s^*(k)} \quad \text{almost surely.}$$

iii) The expected sample size satisfies

$$\frac{\mathbb{E}[N_d|H_k]}{-\log \eta_k} \to \frac{1}{\sum_{s=1}^n q_s \mathcal{D}_s^*(k)}, \quad \text{as } \eta_{\max} \to 0^+. \tag{5}$$

iv) Conditioned on hypothesis H_k, the decision time T_d, as $\eta_{\max} \to 0^+$, satisfies

$$\frac{T_d}{-\log \eta_k} \to \frac{\sum_{s=1}^n q_s \bar{T}_s}{\sum_{s=1}^n q_s \mathcal{D}_s^*(k)} \quad \text{almost surely.}$$

v) The expected decision time satisfies

$$\frac{\mathbb{E}\left[T_d|H_k\right]}{-\log \eta_k} \to \frac{\sum_{s=1}^n q_s \bar{T}_s}{\sum_{s=1}^n q_s \mathcal{D}_s^*(k)} \equiv \frac{\boldsymbol{q} \cdot \boldsymbol{T}}{\boldsymbol{q} \cdot \boldsymbol{\mathcal{D}}^k}, \tag{6}$$

where $\boldsymbol{T}, \boldsymbol{\mathcal{D}}^k \in \mathbb{R}^n_{>0}$ are arrays of mean processing times $\bar{T}_s$ and minimum Kullback-Leibler distances $\mathcal{D}_s^*(k)$.

Proof: We start by establishing the first statement. We let $\eta_{\min} = \min\{\eta_j \mid j \in \{0, \dots, M-1\}\}$. For any fixed $k \in \{0, \dots, M-1\}$, the sample size for decision, denoted by N_d, satisfies

$$N_d \le \left(\text{first } \tau \ge 1 \text{ such that } \sum_{\substack{j=0\\j\neq k}}^{M-1} \prod_{t=1}^{\tau} \frac{f_{s_t}^j(y_t)}{f_{s_t}^k(y_t)} < \eta_{\min}\right)$$
$$\le \left(\text{first } \tau \ge 1 \text{ such that } \prod_{t=1}^{\tau} \frac{f_{s_t}^j(y_t)}{f_{s_t}^k(y_t)} < \frac{\eta_{\min}}{M-1}, \right.$$
$$\left. \text{for all } j \in \{0, \dots, M-1\}, j \neq k\right).$$

Therefore, it follows that

$$\begin{aligned}
&\mathbb{P}(N_d > \tau | H_k) \\
&\le \mathbb{P}\Bigg(\prod_{t=1}^{\tau} \frac{f_{s_t}^j(y_t)}{f_{s_t}^k(y_t)} \ge \frac{\eta_{\min}}{M-1}, j \in \{0, \dots, M-1\} \setminus \{k\} \Bigg| H_k\Bigg) \\
&\le \sum_{\substack{j=0\\j\neq k}}^{M-1} \mathbb{P}\Bigg(\prod_{t=1}^{\tau} \frac{f_{s_t}^j(y_t)}{f_{s_t}^k(y_t)} \ge \frac{\eta_{\min}}{M-1} \Bigg| H_k\Bigg) \\
&= \sum_{\substack{j=0\\j\neq k}}^{M-1} \mathbb{P}\Bigg(\prod_{t=1}^{\tau} \sqrt{\frac{f_{s_t}^j(y_t)}{f_{s_t}^k(y_t)}} \ge \sqrt{\frac{\eta_{\min}}{M-1}} \Bigg| H_k\Bigg) \\
&\le \sum_{\substack{j=0\\j\neq k}}^{M-1} \sqrt{\frac{M-1}{\eta_{\min}}} \mathbb{E}\left[\sqrt{\frac{f_{s^*(j)}^j(Y)}{f_{s^*(j)}^k(Y)}} \Bigg| H_k\right]^{\tau} \qquad (7) \\
&\le \frac{(M-1)^{\frac{3}{2}}}{\sqrt{\eta_{\min}}} \big(\max_{j \in \{0, \dots, M-1\} \setminus \{k\}} \rho_j\big)^{\tau},
\end{aligned}$$

where $s^*(j) = \operatorname{argmax}_{s \in \{1, \dots, n\}} \mathbb{E}\left[\sqrt{\frac{f_s^j(Y)}{f_s^k(Y)}} \Big| H_k\right]$, and

$$\begin{aligned}
\rho_j &= \mathbb{E}\left[\sqrt{\frac{f_{s^*(j)}^j(Y)}{f_{s^*(j)}^k(Y)}} \Bigg| H_k\right] = \int_{\mathbb{R}} \sqrt{f_{s^*(j)}^j(Y) f_{s^*(j)}^k(Y)} dY \\
&< \sqrt{\int_{\mathbb{R}} f_{s^*(j)}^j(Y) dY} \sqrt{\int_{\mathbb{R}} f_{s^*(j)}^k(Y) dY} = 1.
\end{aligned}$$

The inequality (7) follows from the Markov inequality, while $\rho_j < 1$ follows from the Cauchy-Schwarz inequality. Note that the Cauchy-Schwarz inequality is strict because $f_{s^*(j)}^j \neq f_{s^*(j)}^k$ almost surely in $f_{s^*(j)}^k$. To establish almost sure convergence, note that

$$\begin{aligned}
&\sum_{\tau=1}^{\infty} \mathbb{P}(N_d > \tau | H_k) \\
&\qquad \le \sum_{\tau=1}^{\infty} \frac{(M-1)^{\frac{3}{2}}}{\sqrt{\eta_{\min}}} \big(\max_{j \in \{0, \dots, M-1\} \setminus \{k\}} \rho_j\big)^{\tau} < \infty.
\end{aligned}$$

Therefore, by Borel-Cantelli lemma [19], it follows that

$$\mathbb{P}(\limsup_{\tau \to \infty} [N_d > \tau]) = 1 - \mathbb{P}(\liminf_{\tau \to \infty} [N_d \le \tau]) = 0.$$

Thus, for τ large enough, all realizations in the set $\liminf_{\tau \to \infty} [N_d \le \tau]$, converge in finite number of steps. This proves the almost sure convergence on the MSPRT with randomized sensor selection.

To prove the second statement, for hypothesis H_k, let

$$\tilde{N}_d = \left(\text{first } \tau \ge 1 \text{ such that } \sum_{\substack{j=0\\j\neq k}}^{M-1} \prod_{t=1}^{\tau} \frac{f_{s_t}^j(y_t)}{f_{s_t}^k(y_t)} < \eta_k\right),$$

and, accordingly, note that

$$\sum_{\substack{j=0\\j\neq k}}^{M-1} \prod_{t=1}^{\tilde{N}_d - 1} \frac{f_{s_t}^j(y_t)}{f_{s_t}^k(y_t)} \ge \eta_k, \quad \text{and} \quad \sum_{\substack{j=0\\j\neq k}}^{M-1} \prod_{t=1}^{\tilde{N}_d} \frac{f_{s_t}^j(y_t)}{f_{s_t}^k(y_t)} < \eta_k.$$

Some algebraic manipulations on these inequalities yield

$$\begin{aligned}
\frac{-1}{\tilde{N}_d - 1} \log\left(\sum_{\substack{j=0\\j\neq k}}^{M-1} \exp\left(-\sum_{t=1}^{\tilde{N}_d - 1} \log \frac{f_{s_t}^k(y_t)}{f_{s_t}^j(y_t)}\right)\right) &\le \frac{-\log \eta_k}{\tilde{N}_d - 1}, \\
\frac{-1}{\tilde{N}_d} \log\left(\sum_{\substack{j=0\\j\neq k}}^{M-1} \exp\left(-\sum_{t=1}^{\tilde{N}_d} \log \frac{f_{s_t}^k(y_t)}{f_{s_t}^j(y_t)}\right)\right) &> \frac{-\log \eta_k}{\tilde{N}_d}.
\end{aligned} \tag{8}$$

Observe that $\tilde{N}_d \ge N_d$, hence from Lemma 1, $\tilde{N}_d \to \infty$ almost surely as $\eta_{\max} \to 0^+$. In the limit $\tilde{N}_d \to \infty$, the supremum and infimum in inequalities (8) converge to the same value. From Lemma 3, and Lemma 4

$$\begin{aligned}
\frac{1}{\tilde{N}_d} \sum_{t=1}^{\tilde{N}_d} \log \frac{f_{s_t}^k(y_t)}{f_{s_t}^j(y_t)} &\to \frac{1}{\tilde{N}_d} \sum_{t=1}^{\tilde{N}_d} \mathbb{E}\left[\log \frac{f_{s_t}^k(y_t)}{f_{s_t}^j(y_t)} \Bigg| H_k\right] \\
&\to \sum_{s=1}^{n} q_s \mathcal{D}(f_s^k, f_s^j), \text{ almost surely,}
\end{aligned}$$

as $\tilde{N}_d \to \infty$. Lemma 2 implies that the left hand sides of the inequalities (8) almost surely converge to

$$\min_{j\in\{0,\dots,M-1\}\setminus\{k\}} \mathbb{E}\left[\log\frac{f_s^k(Y)}{f_s^j(Y)}\middle|H_k\right] = \sum_{s=1}^n q_s\mathcal{D}_s^*(k).$$

Hence, conditioned on hypothesis H_k

$$\frac{\tilde{N}_d}{-\log\eta_k} \to \frac{1}{\sum_{s=1}^n q_s\mathcal{D}_s^*(k)}$$

almost surely, as $\eta_{\max} \to 0^+$.

Now, notice that

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{N_d}{-\log\eta_k} - \frac{1}{\sum_{s=1}^n q_s\mathcal{D}_s^*(k)}\right| > \epsilon\middle|H_k\right)\\
&= \sum_{v=0}^{M-1}\mathbb{P}\left(\left|\frac{N_d}{-\log\eta_k} - \frac{1}{\sum_{s=1}^n q_s\mathcal{D}_s^*(k)}\right| > \epsilon \;\&\; \text{accept } H_v\middle|H_k\right)\\
&=\mathbb{P}\left(\left|\frac{\tilde{N}_d}{-\log\eta_k} - \frac{1}{\sum_{s=1}^n q_s\mathcal{D}_s^*(k)}\right| > \epsilon\middle|H_k\right)\\
&+\sum_{\substack{v=0\\v\neq k}}^{M-1}\mathbb{P}\left(\left|\frac{N_d}{-\log\eta_k} - \frac{1}{\sum_{s=1}^n q_s\mathcal{D}_s^*(k)}\right| > \epsilon \;\&\; \text{accept } H_v\middle|H_k\right).
\end{aligned}$$

Note that $\alpha_j \to 0^+$, for all $j \in \{0,\dots,M-1\}$, as $\eta_{\max} \to 0^+$. Hence, the right hand side terms above converge to zero as $\eta_{\max} \to 0^+$. This establishes the second statement. We have proved almost sure convergence of $\frac{N_d}{-\log\eta_k}$. To establish convergence in expected value, we construct a Lebesgue integrable upper bound of N_d. Define $\xi_0 = 0$, and for all $m \geq 1$,

$$\xi_m = \Big(\text{first } \tau \geq 1 \text{ such that}$$
$$\sum_{t=\xi_{m-1}+1}^{\xi_{m-1}+\tau}\log\frac{f_{s_t}^k(y_t)}{f_{s_t}^j(y_t)} > 1, \text{ for } j \in \{0,\dots,M-1\}\setminus\{k\}\Big).$$

Note that the variables in the sequence $\{\xi_i\}_{i\in\mathbb{N}}$ are i.i.d., and moreover $\mathbb{E}[\xi_1|H_k] < \infty$, since $\mathcal{D}(f_k^s, f_j^s) > 0$, for all $s \in \{1,\dots,n\}$, and $j \in \{0,\dots,M-1\}\setminus\{k\}$.

Choose $\tilde{\eta} = \lceil\log\frac{M-1}{\eta_k}\rceil$. Note that

$$\sum_{t=1}^{\xi_1+\dots+\xi_{\tilde{\eta}}}\log\frac{f_{s_t}^k(y_t)}{f_{s_t}^j(y_t)} > \tilde{\eta}, \text{ for } j \in \{0,\dots,M-1\}\setminus\{k\}.$$

Hence, $\xi_1 + \dots + \xi_{\tilde{\eta}} \geq N_d$. Further, $\xi_1 + \dots + \xi_{\tilde{\eta}}$ is Lebesgue integrable. The third statement follows from the Lebesgue dominated convergence theorem [19].

To establish the next statement, note that the decision time of MSPRT with randomized sensor selection is the sum of sensor's processing time at each iteration, i.e.,

$$T_d = T_{s_1} + \dots + T_{s_{N_d}}.$$

From Lemma 3, Lemma 1 and Lemma 4, it follows that

$$\frac{T_d}{N_d} \to \frac{1}{N_d}\sum_{t=1}^{N_d}\mathbb{E}[T_{s_t}] \to \sum_{s=1}^n q_s\bar{T}_s,$$

almost surely, as $\eta_{\max} \to 0^+$. Thus, conditioned on hypothesis H_k,

$$\begin{aligned}
\lim_{\eta_{\max}\to 0^+}\frac{T_d}{-\log\eta_k} &= \lim_{\eta_{\max}\to 0^+}\frac{T_d}{N_d}\frac{N_d}{-\log\eta_k}\\
&= \lim_{\eta_{\max}\to 0^+}\frac{T_d}{N_d}\lim_{\eta_{\max}\to 0^+}\frac{N_d}{-\log\eta_k} = \frac{\sum_{s=1}^n q_s\bar{T}_s}{\sum_{s=1}^n q_s\mathcal{D}_s^*(k)},
\end{aligned}$$

almost surely. Now, note that $\{(s_t, T_{s_t})\}_{t\in\mathbb{N}}$ is an i.i.d. realization of the pair (s, T_s). Therefore, by the Wald's identity [19]

$$\mathbb{E}[T_{\xi_1}] = \mathbb{E}\big[\sum_{t=1}^{\xi_1}T_{s_t}\big] = \mathbb{E}[\xi_1]\mathbb{E}[T_s] < \infty.$$

Also, $T_d \leq T_{\xi_1} + \dots + T_{\xi_{\tilde{\eta}}} \in \mathcal{L}^1$. Thus, by the Lebesgue dominated convergence theorem [19]

$$\frac{\mathbb{E}[T_d|H_k]}{-\log\eta_k} \to \frac{\sum_{s=1}^n q_s\bar{T}_s}{\sum_{s=1}^n q_s\mathcal{D}_s^*(k)} = \frac{\boldsymbol{q}\cdot\boldsymbol{T}}{\boldsymbol{q}\cdot\boldsymbol{\mathcal{D}}^k} \quad \text{as } \eta_{\max} \to 0^+.$$

■

Remark 3: The results in Theorem 1 hold if we have at least one sensor with positive minimum Kullback Leibler divergence $\mathcal{D}_s^*(k)$, which is chosen with a positive probability. Thus, the MSPRT with randomized sensor selection is robust to sensor failure and uninformative sensors. In what follows, we assume that at least M sensors are finitely informative. □

Remark 4: In the remainder of the paper, we assume that the error probabilities are chosen small enough, so that the expected decision time is arbitrarily close to the expression in equation (6). □

Remark 5: The MSPRT with randomized sensor selection may not be the optimal sequential test. In fact, this test corresponds to a stationary open-loop strategy. In this paper we wish to determine a time-optimal stationary open-loop strategy, as motivated in Remark 2. □

Remark 6: If the minimum Kullback-Leibler divergence $\mathcal{D}_s^*(k)$ is the same for any given $s \in \{1,\dots,n\}$, and for each $k \in \{0,\dots,M-1\}$, and all thresholds η_k are identical, then the expected decision time is the same conditioned on any hypothesis H_k. For example, if conditioned on hypothesis $H_k, k \in \{0,\dots,M-1\}$, and sensor $s \in \{1,\dots,n\}$, the observation is generated from a Gaussian distribution with mean k and variance σ_s^2, then the minimum Kullback-Leibler divergence from hypothesis k, for sensor s is $\mathcal{D}_s^*(k) = 1/2\sigma_s^2$, which is independent of k. □

V. Optimal sensor selection

In this section we consider sensor selection problems with the aim to minimize the expected decision time of a sequential hypothesis test with randomized sensor selection. As exemplified in Theorem 1, the problem features multiple conditioned decision times and, therefore, multiple distinct cost functions are of interest. In Scenario I below, we aim to minimize the decision time conditioned upon one specific hypothesis being true; in Scenarios II and III we will consider worst-case and average decision times. In all three scenarios the decision variables take values in the probability simplex.

Minimizing decision time conditioned upon a specific hypothesis may be of interest when fast reaction is required in response to the specific hypothesis being indeed true. For example, in change detection problems one aims to quickly detect a change in a stochastic process; the CUSUM algorithm (also referred to as Page's test) [17] is widely used in such problems. It is known [4] that, with fixed threshold, the CUSUM algorithm for quickest change detection is equivalent to an SPRT on the observations taken after the change has occurred. We consider the minimization problem for a single conditioned decision time in Scenario I below and we show that, in this case, observing the best sensor each time is the optimal strategy.

In general, no specific hypothesis might play a special role in the problem and, therefore, it is of interest to simultaneously minimize multiple decision times over the probability simplex. This is a multi-objective optimization problem, and may have Pareto-optimal solutions. We tackle this problem by constructing a single aggregate objective function. In the binary hypothesis case, we construct two single aggregate objective functions as the maximum and the average of the two conditioned decision times. These two functions are discussed in Scenario II and Scenario III respectively. In the multiple hypothesis setting, we consider the single aggregate objective function constructed as the average of the conditioned decision times. An analytical treatment of this function for $M > 2$, is difficult. We determine the optimal number of sensors to be observed, and direct the interested reader to some iterative algorithms to solve such optimization problems. This case is also considered under Scenario III.

Before we pose the problem of optimal sensor selection, we introduce the following notation. We denote the probability simplex in $\mathbb{R}^n$ by Δ_{n-1}, and the vertices of the probability simplex Δ_{n-1} by $\boldsymbol{e}_i$, $i \in \{1, \dots, n\}$. We refer to the line joining any two vertices of the simplex as an *edge*. Finally, we define $g^k : \Delta_{n-1} \to \mathbb{R}$, $k \in \{0, \dots, M-1\}$, by $g^k(\boldsymbol{q}) = \boldsymbol{q} \cdot \boldsymbol{T} / \boldsymbol{q} \cdot \boldsymbol{I}^k$, where $\boldsymbol{I}^k = -\boldsymbol{\mathcal{D}}^k / \log \eta_k$.

A. Scenario I (Optimization of conditioned decision time):

We consider the case when the supervisor is trying to detect a particular hypothesis, irrespective of the present hypothesis. The corresponding optimization problem for a fixed $k \in \{0, \dots, M-1\}$ is posed in the following way:

$$\begin{aligned} \text{minimize} \quad & g^k(\boldsymbol{q}) \\ \text{subject to} \quad & \boldsymbol{q} \in \Delta_{n-1}. \end{aligned} \tag{9}$$

The solution to this minimization problem is given in the following theorem.

Theorem 2 (Optimization of conditioned decision time): The solution to the minimization problem (9) is $\boldsymbol{q}^* = \boldsymbol{e}_{s^*}$, where s^* is given by

$$s^* = \underset{s \in \{1,\dots,n\}}{\operatorname{argmin}} \frac{T_s}{I_s^k},$$

and the minimum objective function is

$$\mathbb{E}\left[T_d^* | H_k\right] = \frac{T_{s^*}}{I_{s^*}^k}. \tag{10}$$

Proof: We notice that objective function is a linear-fractional function. In the following argument, we show that the minima occurs at one of the vertices of the simplex.

We first notice that the probability simplex is the convex hull of the vertices, i.e., any point $\tilde{\boldsymbol{q}}$ in the probability simplex can be written as

$$\tilde{\boldsymbol{q}} = \sum_{s=1}^{n} \alpha_s \boldsymbol{e}_s, \qquad \sum_{s=1}^{n} \alpha_s = 1, \text{ and } \alpha_s \geq 0.$$

We invoke equation (1), and observe that for some $\beta \in [0, 1]$ and for any $s, r \in \{1, \dots, n\}$

$$g^k(\beta \boldsymbol{e}_s + (1-\beta)\boldsymbol{e}_r) \geq \min\{g^k(\boldsymbol{e}_s), g^k(\boldsymbol{e}_r)\}, \tag{11}$$

which can be easily generalized to

$$g^k(\tilde{\boldsymbol{q}}) \geq \min_{s \in \{1,\dots,n\}} g^k(\boldsymbol{e}_s), \tag{12}$$

for any point $\tilde{\boldsymbol{q}}$ in the probability simplex Δ_{n-1}. Hence, minima will occur at one of the vertices $\boldsymbol{e}_{s^*}$, where s^* is given by

$$s^* = \underset{s \in \{1,\dots,n\}}{\operatorname{argmin}} g^k(\boldsymbol{e}_s) = \underset{s \in \{1,\dots,n\}}{\operatorname{argmin}} \frac{T_s}{I_s^k}.$$

■

B. Scenario II (Optimization of the worst case decision time):

For the binary hypothesis testing, we consider the multi-objective optimization problem of minimizing both decision times simultaneously. We construct single aggregate objective function by considering the maximum of the two objective functions. This turns out to be a worst case analysis, and the optimization problem for this case is posed in the following way:

$$\begin{aligned} \text{minimize} \quad & \max\left\{g^0(\boldsymbol{q}), g^1(\boldsymbol{q})\right\}, \\ \text{subject to} \quad & \boldsymbol{q} \in \Delta_{n-1}. \end{aligned} \tag{13}$$

Before we move on to the solution of above minimization problem, we state the following results.

Lemma 5 (Monotonicity of conditioned decision times): The functions g^k, $k \in \{0, \dots, M-1\}$ are monotone on the probability simplex Δ_{n-1}, in the sense that given two points $\boldsymbol{q}_a, \boldsymbol{q}_b \in \Delta_{n-1}$, the function g^k is monotonically non-increasing or monotonically non-decreasing along the line joining $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$.

Proof: Consider probability vectors $\boldsymbol{q}_a, \boldsymbol{q}_b \in \Delta_{n-1}$. Any point $\boldsymbol{q}$ on line joining $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ can be written as $\boldsymbol{q}(\nu) = \nu \boldsymbol{q}_a + (1-\nu)\boldsymbol{q}_b$, $\nu \in]0, 1[$. We note that $g^k(\boldsymbol{q}(\nu))$ is given by:

$$g^k(\boldsymbol{q}(\nu)) = \frac{\nu(\boldsymbol{q}_a \cdot T) + (1-\nu)(\boldsymbol{q}_b \cdot \boldsymbol{T})}{\nu(\boldsymbol{q}_a \cdot \boldsymbol{I}^k) + (1-\nu)(\boldsymbol{q}_b \cdot \boldsymbol{I}^k)}.$$

The derivative of g^k along the line joining $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ is given by

$$\frac{d}{d\nu} g^k(\boldsymbol{q}(\nu)) = \left(g^k(\boldsymbol{q}_a) - g^k(\boldsymbol{q}_b)\right) \times \frac{(\boldsymbol{q}_a \cdot \boldsymbol{I}^k)(\boldsymbol{q}_b \cdot \boldsymbol{I}^k)}{(\nu(\boldsymbol{q}_a \cdot \boldsymbol{I}^k) + (1-\nu)(\boldsymbol{q}_b \cdot \boldsymbol{I}^k))^2}.$$

We note that the sign of the derivative of g^k along the line joining two points $\boldsymbol{q}_a, \boldsymbol{q}_b$ is fixed by the choice of $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$. Hence, the function g^k is monotone over the line joining $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$. Moreover, note that if $g^k(\boldsymbol{q}_a) \neq g^k(\boldsymbol{q}_b)$, then g^k is strictly monotone. Otherwise, g^k is constant over the line joining $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$. ■

Lemma 6 (Location of min-max): Define $g : \Delta_{n-1} \to \mathbb{R}_{\geq 0}$ by $g = \max\{g^0, g^1\}$. A minimum of g lies at the intersection of the graphs of g^0 and g^1, or at some vertex of the probability simplex Δ_{n-1}.

Proof: The idea of the proof is illustrated in Figure 2. We now prove it rigorously.

Case 1: The graphs of g^0 and g^1 do not intersect at any point in the simplex Δ_{n-1}.

In this case, one of the functions g^0 and g^1 is an upper bound to the other function at every point in the probability simplex Δ_{n-1}. Hence, $g = g^k$, for some $k \in \{0, 1\}$, at every point in the probability simplex Δ_{n-1}. From Theorem 2, we know that the minima of g^k on the probability simplex Δ_{n-1} lie at some vertex of the probability simplex Δ_{n-1}.

Case 2: The graphs of g^0 and g^1 intersect at a set Q in the probability simplex Δ_{n-1}, and let $\bar{\boldsymbol{q}}$ be some point in the set Q.

Suppose, a minimum of g occurs at some point $\boldsymbol{q}^* \in \text{relint}(\Delta_{n-1})$, and $\boldsymbol{q}^* \notin Q$, where $\text{relint}(\cdot)$ denotes the relative interior. With out loss of generality, we can assume that $g^0(\boldsymbol{q}^*) > g^1(\boldsymbol{q}^*)$. Also, $g^0(\bar{\boldsymbol{q}}) = g^1(\bar{\boldsymbol{q}})$, and $g^0(\boldsymbol{q}^*) < g^0(\bar{\boldsymbol{q}})$ by assumption.

We invoke Lemma 5, and notice that g^0 and g^1 can intersect at most once on a line. Moreover, we note that $g^0(\boldsymbol{q}^*) > g^1(\boldsymbol{q}^*)$, hence, along the half-line from $\bar{\boldsymbol{q}}$ through $\boldsymbol{q}^*$, $g^0 > g^1$, that is, $g = g^0$. Since $g^0(\boldsymbol{q}^*) < g^0(\bar{\boldsymbol{q}})$, g is decreasing along this half-line. Hence, g should achieve its minimum at the boundary of the simplex Δ_{n-1}, which contradicts that $\boldsymbol{q}^*$ is in the relative interior of the simplex Δ_{n-1}. In summary, if a minimum of g lies in the relative interior of the probability simplex Δ_{n-1}, then it lies at the intersection of the graphs of g^0 and g^1.

The same argument can be applied recursively to show that if a minimum lies at some point $\boldsymbol{q}^\dagger$ on the boundary, then either $g^0(\boldsymbol{q}^\dagger) = g^1(\boldsymbol{q}^\dagger)$ or the minimum lies at the vertex. ■

In the following arguments, let Q be the set of points in the simplex Δ_{n-1}, where $g^0 = g^1$, that is,

$$Q = \{\boldsymbol{q} \in \Delta_{n-1} \mid \boldsymbol{q} \cdot (\boldsymbol{I}^0 - \boldsymbol{I}^1) = 0\}. \tag{14}$$

Also notice that the set Q is non empty if and only if $\boldsymbol{I}^0 - \boldsymbol{I}^1$ has at least one non-negative and one non-positive entry. If the set Q is empty, then it follows from Lemma 6 that the solution of optimization problem in equation (13) lies at some vertex of the probability simplex Δ_{n-1}. Now we consider the case when Q is non empty. We assume that the sensors have been re-ordered such that the entries in $\boldsymbol{I}^0 - \boldsymbol{I}^1$ are in ascending order. We further assume that, for $\boldsymbol{I}^0 - \boldsymbol{I}^1$, the first m entries, $m < n$, are non positive, and the remaining entries are positive.

Lemma 7 (Intersection polytope): If the set Q defined in equation (14) is non empty, then the polytope generated by the points in the set Q has vertices given by:

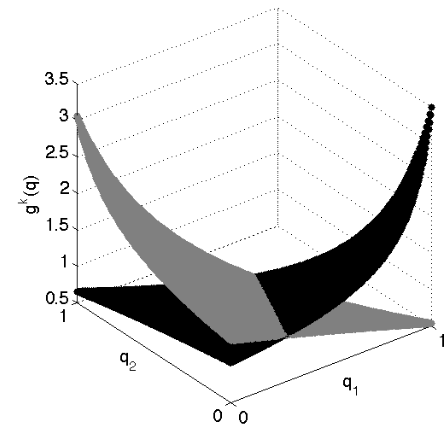

Fig. 2. Linear-fractional functions. Both the functions achieve their minima at some vertex of the simplex. The maximum of the two functions achieves its minimum at the intersection of two graphs.

$$\tilde{Q} = \{\tilde{\boldsymbol{q}}^{sr} \mid s \in \{1, \ldots, m\} \text{ and } r \in \{m+1, \ldots, n\}\},$$

where for each $i \in \{1, \ldots, n\}$

$$\tilde{q}_i^{sr} = \begin{cases} \dfrac{(I_r^0 - I_r^1)}{(I_r^0 - I_r^1) - (I_s^0 - I_s^1)}, & \text{if } i = s, \\ 1 - \tilde{q}_s^{sr}, & \text{if } i = r, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Proof: Any $q \in Q$ satisfies the following constraints

$$\sum_{s=1}^{n} q_s = 1, \quad q_s \in [0, 1], \tag{16}$$

$$\sum_{s=1}^{n} q_s (I_s^0 - I_s^1) = 0, \tag{17}$$

Eliminating q_n, using equation (16) and equation (17), we get:

$$\sum_{s=1}^{n-1} \beta_s q_s = 1, \text{ where } \beta_s = \frac{(I_n^0 - I_n^1) - (I_s^0 - I_s^1)}{(I_n^0 - I_n^1)}. \tag{18}$$

The equation (18) defines a hyperplane, whose extreme points in $\mathbb{R}_{\geq 0}^{n-1}$ are given by

$$\tilde{\boldsymbol{q}}^{sn} = \frac{1}{\beta_s} \boldsymbol{e}_s, \quad s \in \{1, \ldots, n-1\}.$$

Note that for $s \in \{1, \ldots, m\}$, $\tilde{\boldsymbol{q}}^{sn} \in \Delta_{n-1}$. Hence, these points define some vertices of the polytope generated by points in the set Q. Also note that the other vertices of the polytope can be determined by the intersection of each pair of lines through $\tilde{\boldsymbol{q}}^{sn}$ and $\tilde{\boldsymbol{q}}^{rn}$, and $\boldsymbol{e}_s$ and $\boldsymbol{e}_r$, for $s \in \{1, \ldots, m\}$, and $r \in \{m+1, \ldots, n-1\}$. In particular, these vertices are given by $\tilde{\boldsymbol{q}}^{sr}$ defined in equation (15).

Hence, all the vertices of the polytopes are defined by $\tilde{\boldsymbol{q}}^{sr}$, $s \in \{1, \ldots, m\}$, $r \in \{m+1, \ldots, n\}$. Therefore, the set of vertices of the polygon generated by the points in the set Q is $\tilde{Q}$. ■

Before we state the solution to the optimization problem (13), we define the following:

$$(s^*,r^*) \in \operatorname*{argmin}_{\substack{r\in\{m+1,\dots,n\}\\ s\in\{1,\dots,m\}}} \frac{(I_r^0-I_r^1)T_s-(I_s^0-I_s^1)T_r}{I_s^1I_r^0-I_s^0I_r^1}, \quad \text{and}$$

$$g_{\text{two-sensors}}(s^*,r^*) = \frac{(I_{r^*}^0-I_{r^*}^1)T_{s^*}-(I_{s^*}^0-I_{s^*}^1)T_{r^*}}{I_{s^*}^1I_{r^*}^0-I_{s^*}^0I_{r^*}^1}.$$

We also define

$$w^* = \operatorname*{argmin}_{w\in\{1,\dots,n\}} \max\left\{\frac{T_w}{I_w^0},\frac{T_w}{I_w^1}\right\}, \text{ and}$$

$$g_{\text{one-sensor}}(w^*) = \max\left\{\frac{T_{w^*}}{I_{w^*}^0},\frac{T_{w^*}}{I_{w^*}^1}\right\}.$$

Theorem 3 (Worst case optimization): For the optimization problem (13), an optimal probability vector is given by:

$$\boldsymbol{q}^* = \begin{cases} \boldsymbol{e}_{w^*}, & \text{if } g_{\text{one-sensor}}(w^*) \le g_{\text{two-sensors}}(s^*,r^*),\\ \tilde{\boldsymbol{q}}^{s^*r^*}, & \text{if } g_{\text{one-sensor}}(w^*) > g_{\text{two-sensors}}(s^*,r^*),\end{cases}$$

and the minimum value of the function is given by:

$$\min\left\{g_{\text{one-sensor}}(w^*), g_{\text{two-sensors}}(s^*,r^*)\right\}.$$

Proof: We invoke Lemma 6, and note that a minimum should lie at some vertex of the simplex Δ_{n-1}, or at some point in the set Q. Note that $g^0=g^1$ on the set Q, hence the problem of minimizing $\max\{g^0,g^1\}$ reduces to minimizing g^0 on the set Q. From Theorem 2, we know that g^0 achieves the minima at some extreme point of the feasible region. From Lemma 7, we know that the vertices of the polytope generated by points in set Q are given by set $\tilde{Q}$. We further note that $g_{\text{two-sensors}}(s,r)$ and $g_{\text{one-sensor}}(w)$ are the value of objective function at the points in the set $\tilde{Q}$ and the vertices of the probability simplex Δ_{n-1} respectively, which completes the proof. ■

C. Scenario III (Optimization of the average decision time):

For the multi-objective optimization problem of minimizing all the decision times simultaneously on the simplex, we formulate the single aggregate objective function as the average of these decision times. The resulting optimization problem, for $M\ge 2$, is posed in the following way:

$$\begin{aligned} &\text{minimize} && \frac{1}{M}(g^0(\boldsymbol{q})+\dots+g^{M-1}(\boldsymbol{q})),\\ &\text{subject to} && \boldsymbol{q}\in\Delta_{n-1}.\end{aligned} \tag{19}$$

In the following discussion we assume $n>M$, unless otherwise stated. We analyze the optimization problem in equation (19) as follows:

Lemma 8 (Non-vanishing Jacobian): The objective function in optimization problem in equation (19) has no critical point on Δ_{n-1} if the vectors $\boldsymbol{T},\boldsymbol{I}^0,\dots,\boldsymbol{I}^{M-1}\in\mathbb{R}^n_{>0}$ are linearly independent.

Proof: The Jacobian of the objective function in the optimization problem in equation (19) is

$$\frac{1}{M}\frac{\partial}{\partial \boldsymbol{q}}\sum_{k=0}^{M-1} g^k = \Gamma\psi(\boldsymbol{q}),$$

where $\Gamma = \frac{1}{M}\begin{bmatrix}\boldsymbol{T} & -\boldsymbol{I}^0 & \dots & -\boldsymbol{I}^{M-1}\end{bmatrix}\in\mathbb{R}^{n\times(M+1)}$, and $\psi:\Delta_{n-1}\to\mathbb{R}^{M+1}$ is defined by

$$\psi(\boldsymbol{q}) = \begin{bmatrix}\displaystyle\sum_{k=0}^{M-1}\frac{1}{\boldsymbol{q}\cdot\boldsymbol{I}^k} & \dfrac{\boldsymbol{q}\cdot\boldsymbol{T}}{(\boldsymbol{q}\cdot\boldsymbol{I}^0)^2} & \dots & \dfrac{\boldsymbol{q}\cdot\boldsymbol{T}}{(\boldsymbol{q}\cdot\boldsymbol{I}^{M-1})^2}\end{bmatrix}^{\mathrm{T}}.$$

For $n>M$, if the vectors $\boldsymbol{T},\boldsymbol{I}^0,\dots,\boldsymbol{I}^{M-1}$ are linearly independent, then Γ is full rank. Further, the entries of ψ are non-zero on the probability simplex Δ_{n-1}. Hence, the Jacobian does not vanish anywhere on the probability simplex Δ_{n-1}. ■

Lemma 9 (Case of Independent Information): For $M=2$, if $\boldsymbol{I}^0$ and $\boldsymbol{I}^1$ are linearly independent, and $\boldsymbol{T}=\alpha_0\boldsymbol{I}^0+\alpha_1\boldsymbol{I}^1$, for some $\alpha_0,\alpha_1\in\mathbb{R}$, then the following statements hold:

i) if α_0 and α_1 have opposite signs, then g^0+g^1 has no critical point on the simplex Δ_{n-1}, and
ii) for $\alpha_0,\alpha_1>0$, g^0+g^1 has a critical point on the simplex Δ_{n-1} if and only if there exists $v\in\Delta_{n-1}$ perpendicular to the vector $\sqrt{\alpha_0}\boldsymbol{I}^0-\sqrt{\alpha_1}\boldsymbol{I}^1$.

Proof: We notice that the Jacobian of g^0+g^1 satisfies

$$\begin{aligned}(\boldsymbol{q}\cdot\boldsymbol{I}^0)^2(\boldsymbol{q}\cdot\boldsymbol{I}^1)^2&\frac{\partial}{\partial\boldsymbol{q}}(g^0+g^1)\\ &=\boldsymbol{T}\left((\boldsymbol{q}\cdot\boldsymbol{I}^0)(\boldsymbol{q}\cdot\boldsymbol{I}^1)^2+(\boldsymbol{q}\cdot\boldsymbol{I}^1)(\boldsymbol{q}\cdot\boldsymbol{I}^0)^2\right)\\ &\quad-\boldsymbol{I}^0(\boldsymbol{q}\cdot\boldsymbol{T})(\boldsymbol{q}\cdot\boldsymbol{I}^1)^2-\boldsymbol{I}^1(\boldsymbol{q}\cdot\boldsymbol{T})(\boldsymbol{q}\cdot\boldsymbol{I}^0)^2.\end{aligned} \tag{20}$$

Substituting $\boldsymbol{T}=\alpha_0\boldsymbol{I}^0+\alpha_1\boldsymbol{I}^1$, equation (20) becomes

$$\begin{aligned}&(\boldsymbol{q}\cdot\boldsymbol{I}^0)^2(\boldsymbol{q}\cdot\boldsymbol{I}^1)^2\frac{\partial}{\partial\boldsymbol{q}}(g^0+g^1)\\ &=\left(\alpha_0(\boldsymbol{q}\cdot\boldsymbol{I}^0)^2-\alpha_1(\boldsymbol{q}\cdot\boldsymbol{I}^1)^2\right)\left((\boldsymbol{q}\cdot\boldsymbol{I}^1)\boldsymbol{I}^0-(\boldsymbol{q}\cdot\boldsymbol{I}^0)\boldsymbol{I}^1\right).\end{aligned}$$

Since $\boldsymbol{I}^0$, and $\boldsymbol{I}^1$ are linearly independent, we have

$$\frac{\partial}{\partial\boldsymbol{q}}(g^0+g^1)=0 \iff \alpha_0(\boldsymbol{q}\cdot\boldsymbol{I}^0)^2-\alpha_1(\boldsymbol{q}\cdot\boldsymbol{I}^1)^2=0.$$

Hence, g^0+g^1 has a critical point on the simplex Δ_{n-1} if and only if

$$\alpha_0(\boldsymbol{q}\cdot\boldsymbol{I}^0)^2=\alpha_1(\boldsymbol{q}\cdot\boldsymbol{I}^1)^2. \tag{21}$$

Notice that, if α_0, and α_1 have opposite signs, then equation (21) can not be satisfied for any $\boldsymbol{q}\in\Delta_{n-1}$, and hence, g^0+g^1 has no critical point on the simplex Δ_{n-1}.

If $\alpha_0,\alpha_1>0$, then equation (21) leads to

$$\boldsymbol{q}\cdot(\sqrt{\alpha_0}\boldsymbol{I}^0-\sqrt{\alpha_1}\boldsymbol{I}^1)=0.$$

Therefore, g^0+g^1 has a critical point on the simplex Δ_{n-1} if and only if there exists $v\in\Delta_{n-1}$ perpendicular to the vector $\sqrt{\alpha_0}\boldsymbol{I}^0-\sqrt{\alpha_1}\boldsymbol{I}^1$. ■

Lemma 10 (Optimal number of sensors): For $n>M$, if each $(M+1)\times(M+1)$ sub-matrix of the matrix

$$\Gamma=\begin{bmatrix}\boldsymbol{T} & -\boldsymbol{I}^0 & \dots & -\boldsymbol{I}^{M-1}\end{bmatrix}\in\mathbb{R}^{n\times(M+1)}$$

is full rank, then the following statements hold:

i) every solution of the optimization problem (19) lies on the probability simplex $\Delta_{M-1} \subset \Delta_{n-1}$; and
ii) every time-optimal policy requires at most M sensors to be observed.

Proof: From Lemma 8, we know that if $\boldsymbol{T}, \boldsymbol{I}^0, \dots, \boldsymbol{I}^{M-1}$ are linearly independent, then the Jacobian of the objective function in equation (19) does not vanish anywhere on the simplex Δ_{n-1}. Hence, a minimum lies at some simplex Δ_{n-2}, which is the boundary of the simplex Δ_{n-1}. Notice that, if $n > M$ and the condition in the lemma holds, then the projections of $\boldsymbol{T}, \boldsymbol{I}^0, \dots, \boldsymbol{I}^{M-1}$ on the simplex Δ_{n-2} are also linearly independent, and the argument repeats. Hence, a minimum lies at some simplex Δ_{M-1}, which implies that optimal policy requires at most M sensors to be observed. ■

Lemma 11 (Optimization on an edge): Given two vertices $\boldsymbol{e}_s$ and $\boldsymbol{e}_r$, $s \neq r$, of the probability simplex Δ_{n-1}, then for the objective function in the problem (19) with $M = 2$, the following statements hold:

i) if $g^0(\boldsymbol{e}_s) < g^0(\boldsymbol{e}_r)$, and $g^1(\boldsymbol{e}_s) < g^1(\boldsymbol{e}_r)$, then the minima, along the edge joining $\boldsymbol{e}_s$ and $\boldsymbol{e}_r$, lies at $\boldsymbol{e}_s$, and optimal value is given by $\frac{1}{2}(g^0(\boldsymbol{e}_s) + g^1(\boldsymbol{e}_s))$; and
ii) if $g^0(\boldsymbol{e}_s) < g^0(\boldsymbol{e}_r)$, and $g^1(\boldsymbol{e}_s) > g^1(\boldsymbol{e}_r)$, then the minima, along the edge joining $\boldsymbol{e}_s$ and $\boldsymbol{e}_r$, lies at the point $\boldsymbol{q}^* = (1-\nu^*)\boldsymbol{e}_s + \nu^*\boldsymbol{e}_r$, where

$$\nu^* = \min\Big\{1, \Big(\frac{1}{1+\mu}\Big)^+\Big\},$$
$$\mu = \frac{I_r^0\sqrt{T_sI_r^1 - T_rI_s^1} - I_r^1\sqrt{T_rI_s^0 - T_sI_r^0}}{I_s^1\sqrt{T_rI_s^0 - T_sI_r^0} - I_s^0\sqrt{T_sI_r^1 - T_rI_s^1}},$$

and the optimal value is given by

$$\begin{cases} \frac{1}{2}(g^0(\boldsymbol{e}_s) + g^1(\boldsymbol{e}_s)), & \text{if } \nu^* = 0, \\ \frac{1}{2}(g^0(\boldsymbol{e}_r) + g^1(\boldsymbol{e}_r)), & \text{if } \nu^* = 1, \\ \frac{1}{2}\left(\sqrt{\frac{T_sI_r^1 - T_rI_s^1}{I_s^0I_r^1 - I_r^0I_s^1}} + \sqrt{\frac{T_rI_s^0 - T_sI_r^0}{I_s^0I_r^1 - I_r^0I_s^1}}\right)^2, & \text{otherwise.} \end{cases}$$

Proof: We observe from Lemma 5 that both g^0, and g^1 are monotonically non-increasing or non-decreasing along any line. Hence, if $g^0(\boldsymbol{e}_s) < g^0(\boldsymbol{e}_r)$, and $g^1(\boldsymbol{e}_s) < g^1(\boldsymbol{e}_r)$, then the minima should lie at $\boldsymbol{e}_s$. This concludes the proof of the first statement. We now establish the second statement. We note that any point on the line segment connecting $\boldsymbol{e}_s$ and $\boldsymbol{e}_r$ can be written as $\boldsymbol{q}(\nu) = (1-\nu)\boldsymbol{e}_s + \nu\boldsymbol{e}_r$. The value of $g^0 + g^1$ at $\boldsymbol{q}$ is

$$g^0(\boldsymbol{q}(\nu)) + g^1(\boldsymbol{q}(\nu)) = \frac{(1-\nu)T_s + \nu T_r}{(1-\nu)I_s^0 + \nu I_r^0} + \frac{(1-\nu)T_s + \nu T_r}{(1-\nu)I_s^1 + \nu I_r^1}.$$

Differentiating with respect to ν, we get

$$g^{0\prime}(\boldsymbol{q}(\nu)) + g^{1\prime}(\boldsymbol{q}(\nu)) = \frac{I_s^0T_r - T_sI_r^0}{(I_s^0 + \nu(I_r^0 - I_s^0))^2} + \frac{I_s^1T_r - T_sI_r^1}{(I_s^1 + \nu(I_r^1 - I_s^1))^2}. \quad (22)$$

Notice that the two terms in equation (22) have opposite sign. Setting the derivative to zero, and choosing the value of ν in $[0, 1]$, we get $\nu^* = \min\{1, 1/(1+\mu)^+\}$, where μ is as defined in the statement of the theorem. The optimal value of the function can be obtained, by substituting $\nu = \nu^*$ in the expression for $\frac{1}{2}(g^0(\boldsymbol{q}(\nu)) + g^1(\boldsymbol{q}(\nu)))$. ■

Theorem 4 (Optimization of average decision time): For the optimization problem (19) with $M = 2$, the following statements hold:

i) If $\boldsymbol{I}^0$, $\boldsymbol{I}^1$ are linearly dependent, then the solution lies at some vertex of the simplex Δ_{n-1}.
ii) If $\boldsymbol{I}^0$ and $\boldsymbol{I}^1$ are linearly independent, and $\boldsymbol{T} = \alpha_0\boldsymbol{I}^0 + \alpha_1\boldsymbol{I}^1$, $\alpha_0, \alpha_1 \in \mathbb{R}$, then the following statements hold:
    a) If α_0 and α_1 have opposite signs, then the optimal solution lies at some edge of the simplex Δ_{n-1}.
    b) If $\alpha_0, \alpha_1 > 0$, then the optimal solution may lie in the interior of the simplex Δ_{n-1}.
iii) If every 3×3 sub-matrix of the matrix $\begin{bmatrix}\boldsymbol{T} & \boldsymbol{I}^0 & \boldsymbol{I}^1\end{bmatrix} \in \mathbb{R}^{n\times 3}$ is full rank, then a minimum lies at an edge of the simplex Δ_{n-1}.

Proof: We start by establishing the first statement. Since, $\boldsymbol{I}^0$ and $\boldsymbol{I}^1$ are linearly dependent, there exists a $\gamma > 0$ such that $\boldsymbol{I}^0 = \gamma\boldsymbol{I}^1$. For $\boldsymbol{I}^0 = \gamma\boldsymbol{I}^1$, we have $g^0 + g^1 = (1+\gamma)g^0$. Hence, the minima of $g^0 + g^1$ lies at the same point where g^0 achieves the minima. From Theorem 2, it follows that g^0 achieves the minima at some vertex of the simplex Δ_{n-1}.

To prove the second statement, we note that from Lemma 9, it follows that if α_0, and α_1 have opposite signs, then the Jacobian of $g^0 + g^1$ does not vanish anywhere on the simplex Δ_{n-1}. Hence, the minima lies at the boundary of the simplex. Notice that the boundary, of the simplex Δ_{n-1}, are n simplices Δ_{n-2}. Notice that the argument repeats till $n > 2$. Hence, the optima lie on one of the $\binom{n}{2}$ simplices Δ_1, which are the edges of the original simplex. Moreover, we note that from Lemma 9, it follows that if $\alpha_0, \alpha_1 > 0$, then we can not guarantee the number of optimal set of sensors. This concludes the proof of the second statement.

To prove the last statement, we note that it follows immediately from Lemma 10 that a solution of the optimization problem in equation (19) would lie at some simplex Δ_1, which is an edge of the original simplex. ■

Note that, we have shown that, for $M = 2$ and a generic set of sensors, the solution of the optimization problem in equation (19) lies at an edge of the simplex Δ_{n-1}. The optimal value of the objective function on a given edge was determined in Lemma 11. Hence, an optimal solution of this problem can be determined by a comparison of the optimal values at each edge.

For the multiple hypothesis case, we have determined the time-optimal number of the sensors to be observed in Lemma 10. In order to identify these sensors, one needs to solve the optimization problem in equation (19). We notice that the objective function in this optimization problem is non-convex, and is hard to tackle analytically for $M > 2$. Interested reader may refer to some efficient iterative algorithms in linear-fractional programming literature (e.g., [6]) to solve these problems.

VI. Numerical Examples

We now elucidate on the results obtained in previous sections through some numerical examples. We present three examples, which provide further insights into the scenarios considered in Section V. In the first one, we consider four sensors with ternary outputs, and three hypotheses. We compare the conditioned asymptotic decision times, obtained in Theorem 1, with the decision times obtained numerically through Monte-Carlo simulations. In the second example, for the same set of sensors and hypothesis, we compare the optimal average decision time, obtained in Theorem 4, with some particular average decision times. In the third example, we compare the worst case optimal decision time obtained in Theorem 3 with some particular worst-case expected decision times.

Example 1 (Conditional expected decision time): We consider four sensors connected to a fusion center, and three underlying hypothesis. We assume that the sensors take ternary measurements $\{0, 1, 2\}$. The probabilities of their measurement being zero and one, under three hypotheses, are randomly chosen and are shown in Tables I and II, respectively. The probability of measurement being two is obtained by subtracting these probabilities from one. The processing times on the sensors are randomly chosen to be 0.68, 3.19, 5.31, and 6.55 seconds, respectively.

TABLE I
CONDITIONAL PROBABILITIES OF MEASUREMENT BEING ZERO

Sensor	Probability(0)		
	Hypothesis 0	Hypothesis 1	Hypothesis 2
1	0.4218	0.2106	0.2769
2	0.9157	0.0415	0.3025
3	0.7922	0.1814	0.0971
4	0.9595	0.0193	0.0061

TABLE II
CONDITIONAL PROBABILITIES OF MEASUREMENT BEING ONE

Sensor	Probability(1)		
	Hypothesis 0	Hypothesis 1	Hypothesis 2
1	0.1991	0.6787	0.2207
2	0.0813	0.7577	0.0462
3	0.0313	0.7431	0.0449
4	0.0027	0.5884	0.1705

We performed Monte-Carlo simulations to numerically obtain the expected decision time, conditioned on hypothesis H_0. For different sensor selection probabilities, a comparison of the numerically obtained expected decision times with the theoretical expected decision times is shown in Figure 3. These results suggest that the asymptotic decision times obtained in Theorem 1 provide a lower bound to the conditional expected decision times for the larger error probabilities. It can be seen from Figure 3, and verified from Theorem 2 that conditioned on hypothesis H_0, sensor 4 is the optimal sensor. Notice the processing time and information trade-off. Despite having the highest processing time, conditioned on hypothesis H_0, the sensor 4 is optimal. This is due to the fact that sensor 4 is highly informative on hypothesis H_0.

Example 2 (Optimal average expected decision time): For the same set of data in Example 1, we now determine the optimal policies for the average expected decision time. A comparison of average expected decision time for different sensor selection probabilities is shown in Figure 4. An optimal

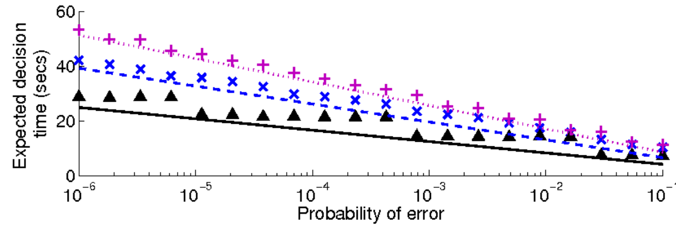

Fig. 3. Expected decision time conditioned on hypothesis H_0 plotted on semi-log axes. The dotted magenta line and magenta "+" represent the theoretical and numerical expected decision time for average expected decision time-optimal sensor selection policy, respectively. The dashed blue line and blue "$\times$" represent the theoretical and numerical expected decision time for the uniform sensor selection policy, respectively. The solid black line and black triangles represent the theoretical and numerical expected decision time when only optimal sensor 4 is selected.

average expected decision time sensor selection probability distribution is $\boldsymbol{q} = [0\ 0.98\ 0\ 0.02]$. It can be seen that the optimal policy significantly improves the average expected decision time over the uniform policy. The sensor 4 which is the optimal sensor conditioned on hypothesis H_0 is now chosen with a very small probability. This is due to the poor performance of the sensor 4 under hypothesis H_1 and H_2 and its high processing time. Good performance under one hypothesis and poor performance under other hypothesis is common in weather-sensitive sensors, e.g., sensor performs extremely well in sunny conditions, but in cloudy or rainy conditions its performance deteriorates significantly.

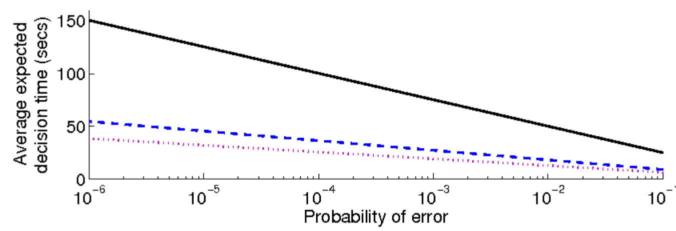

Fig. 4. Average expected decision times plotted on semi-log axes. The black solid line represents the policy where only sensor 4 is selected. The blue dashed line represents the uniform sensor selection policy. The magenta dotted line is average expected decision time-optimal policy.

Example 3 (Optimal worst case decision time): For the same set of data in Example 1, we now determine the optimal policies for the average expected decision time. For this data, the optimal worst-case sensor selection probability distribution is $\boldsymbol{q} = [0\ 0.91\ 0\ 0.09]$. A comparison of the optimal worst case expected decision time with some particular worst case decision times is shown in Figure 5. It may be verified that for the optimal sensor selection probabilities, the expected decision time, conditioned on hypothesis H_0 and H_2 are the same. This suggests that even for more that two hypothesis, the optimal policy may lie at the intersection of the graphs of the expected decision times.

Remark 7: The optimal results we obtained, may only be sub-optimal because of the asymptotic approximations in equations (6). We further note that, for small error probabilities and large sample sizes, these asymptotic approximations yield fairly accurate results [5], and in fact, this is the regime in which it is of interest to minimize the expected decision

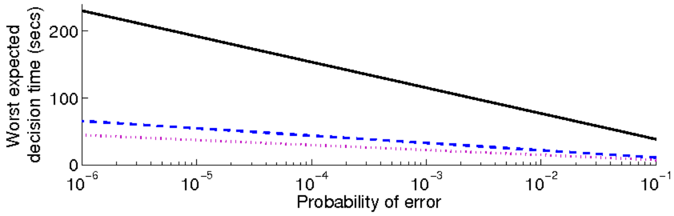

Fig. 5. Worst case expected decision times plotted on semi-log axes. The black solid line represents the policy where only sensor 4 is selected. The blue dashed line represents the uniform sensor selection policy. The magenta dotted line is worst expected decision time-optimal policy.

time. Therefore, for all practical purposes the obtained optimal scheme is very close to the actual optimal scheme. □

VII. Conclusions

In this paper, we considered a sequential decision making problem with randomized sensor selection. We developed version of the MSPRT algorithm where the sensor switches at each observation. We used this sequential procedure to decide reliably. We studied the set of optimal sensors to be observed in order to decide in minimum time. We observed the trade off between the information carried by a sensor and its processing time. A randomized sensor selection strategy was adopted. It was shown that, conditioned on a hypothesis, only one sensor is optimal. Indeed, if the true hypothesis is not known beforehand, then a randomized strategy is justified. For the binary hypothesis case, three performance metrics were considered and it was found that for a generic set of sensors at most two sensors are optimal. Further, it was shown that for M underlying hypotheses, and a generic set of sensors, an optimal policy requires at most M sensors to be observed. It was observed that the optimal set of the sensors is not necessarily the set of optimal sensors conditioned on each hypothesis. A procedure for the identification of the optimal sensors was developed. In the binary hypothesis case, the computational complexity of the procedure for the three scenarios, namely, the conditioned decision time, the worst case decision time, and the average decision time, was $\mathcal{O}(n)$, $\mathcal{O}(n^2)$, and $\mathcal{O}(n^2)$, respectively.

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