PROBABILITY DISTRIBUTION OF STATISTICAL ENERGY ANALYSIS MODEL RESPONSES DUE TO PARAMETER RANDOMNESS

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ABSTRACT

The Statistical Energy Analysis (SEA) methodology has been widely used in aerospace, ship and automotive industry for high frequency noise analysis and acoustic designs. SEA models are treated here as baseline representations of a population of models for systems such as automotive vehicles. SEA responses from the population of all possible models for a vehicle have a random distribution because of the unavoidable uncertainty in the physical parameters due to fabrication imperfection, manufacturing and assembly variations. The random characteristics of the SEA responses can be described by the response probability distribution. In this work, SEA energy response probability distributions due to parameter randomness in frequency bands are proven through the Central Limit Theorem to be Gaussian for infinite number of design parameters. Mean squared sound pressure and velocity are directly proportional to SEA energy responses, their distributions are also shown to be Gaussian. In engineering applications, the number of design parameters is always finite for any SEA models. A Monte Carlo test and Statistical Hypothesis test on a simple 3-element SEA model show that the theoretical, infinite order, Gaussian distributions are good approximations for response distributions of a finite parameter SEA model.

INTRODUCTION

Quality acoustic designs for aircraft, ships, and automotive vehicles require the mean-rms and the extreme values of the vibro-acoustic responses. Damage from vibration and loudness for acoustic design are a result of extreme response values. Typical SEA model predictions include only the mean-rms energy responses of a system. Variance and the probability analysis of extreme vibro-acoustic response values provide information for worst case system response scenarios. The probability distributions of SEA responses determine the response variances and confidence levels needed to evaluate the probability of excessive acoustic noise and structural vibration (Lyon, 1987).

Study of variance and confidence level of SEA responses was initiated by Lyon (1967, 1975). In his study, Gamma distributions were chosen to compute confidence levels because they have the desirable feature of strictly positive values. SEA response variance analysis for automotive vehicle was investigated by DeJong (1985). An exceptionally large experimental study (156 production cars) of vibro-acoustic response statistics was conducted by Kompella and Bernhard (1993). Their experimental results show that the vibration responses of nominally identical structures differ greatly. The statistical fluctuations occurring in the vibration energy flow characteristics of a one-dimensional system was determined by Mahonar and Keane (1994) as a part of their long term study into the reliability of SEA methods. There is no way to model the multi-dimensional joint probability distribution of the large number of parameters and variables by a Monte Carlo procedure (Fahy, 1994). Log-normal distribution of SEA model responses was proposed based on the Central Limit Theory without proof (Lyon and DeJong, 1995).

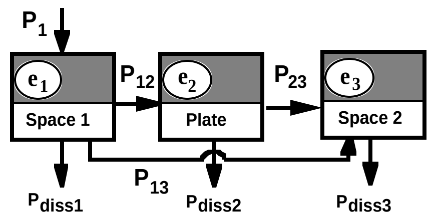

Figure 1: Conceptual SEA Model

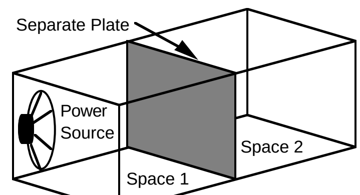

Figure 2: 3-Element SEA Model Schematic

STATISTICAL ENERGY ANALYSIS

SEA can be illustrated with a simple SEA model: two spaces are separated by an aluminum plate (Figure 1 and 2). Stored energy per mode is represented by e_1, e_2 and e_3 in the element blocks. Power flows between the elements are shown by the arrows labeled P_{12}, P_{13}, and P_{23} where the subscripts refer to the elements connected. The model includes one external power source, P_1. Power dissipated by damping and acoustic treatment is represented by P_{diss1}, P_{diss2} and P_{diss3}.

SEA model analysis is a set of simultaneous, frequency dependent, algebraic equations for total element energies.

$$\mathbf{N}(f_k,\mathbf{x})\,\mathbf{e}(f_k,\mathbf{x}) = \frac{1}{2\pi f_k}\mathbf{P}(f_k) \qquad (1)$$

where: $\mathbf{x} = [x_1 \quad \cdots \quad x_M]$, the vector of SEA design parameters
f_k is the kth analysis band center frequency (Hz.)
$\mathbf{e}(f_k,\mathbf{x})$ is a modal energy vector (Joule Hz/mode),
$\mathbf{P}(f_k) = [P_1(f_k) \quad \cdots \quad P_K(f_k)]^T$, input power vector (Watt),
$\mathbf{N}(f_k,\mathbf{x})$ is the SEA system matrix,
$\eta_i \quad = \eta_i(f_k,\mathbf{x})$, internal loss factors (ND), and
$\eta_{ij} \quad = \eta_{ij}(f_k,\mathbf{x})$, coupling loss factor (ND).

The matrix, $\mathbf{N}(f_k,\mathbf{x})$ is a K xK symmetric, diagonally dominant and definite positive matrix.

$$\mathbf{N}\left(f_k,\mathbf{x}\right)=\begin{bmatrix} n_1\eta_1+\sum\limits_{j=2}^{K} n_1\eta_{1j} & \cdots & -n_1\eta_{1i} & \cdots & -n_1\eta_{1K} \\ \vdots & \ddots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ -n_1\eta_{1i} & \cdots & n_i\eta_i+\sum\limits_{j=1}^{i-1} n_j\eta_{ji}+\sum\limits_{j=i+1}^{K} n_i\eta_{ij} & \cdots & -n_i\eta_{iK} \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \ddots & \vdots \\ -n_1\eta_{1K} & \cdots & -n_i\eta_{iK} & \cdots & n_K\eta_K+\sum\limits_{j=1}^{K-1} n_j\eta_{jK} \end{bmatrix} \quad (2)$$

where: $n_i = n_i(f_k,\mathbf{x})$, the element modal density (modes/Hz).

The fundamental assumption of SEA is that the energies in all independent modes equalize at steady state within each, narrow, frequency band. Total energy in "i^{th}" element over "k^{th}" frequency band, $E_{i,k}(f_k,\mathbf{x})$, is derived from the product of the element's modal energy, $e_{i,k}(f_k,\mathbf{x})$ and modal density, $n_i(f_k,\mathbf{x})$.

$$E_{i,k}(f_k,\mathbf{x}) = n_i(f_k,\mathbf{x})e_{i,k}(f_k,\mathbf{x}) \quad (3)$$

The energy in the "i^{th}" element in the "k^{th}" frequency band is related to mean-squared pressure or velocity.

$$E_{i,k}(f_k,\mathbf{x}) = \begin{cases} \dfrac{V_i}{\rho_i c_i^2}\left\langle p_{i,k}^2(f_k,\mathbf{x})\right\rangle & \text{for acoustic spaces} \\ m_i\left\langle v_{i,k}^2(f_k,\mathbf{x})\right\rangle & \text{for structures} \end{cases} \quad (4)$$

where: V_i is acoustic volume (m^3), ρ_i is mass density (kg/m^3),
c_i is speed of sound in medium(m/s), m_i is mass (kg),
$\left\langle p_{i,k}^2\right\rangle$ is expected value of ms sound pressure (Pa^2), and
$\left\langle v_{i,k}^2\right\rangle$ is expected value of mean-squared velocity (m^2/s^2).

Both mean-squared (total) acoustic pressure and mean-squared (total) velocity are the sums of independent frequency components in each frequency band.

$$\begin{cases} \left\langle p_i^2(\mathbf{x})\right\rangle = \sum\limits_{k=1}^{N_f}\left\langle p_{i,k}^2(f_k,\mathbf{x})\right\rangle = \dfrac{\rho_i c_i^2}{V_i}\sum\limits_{k=1}^{N_f} E_{i,k}(f_k,\mathbf{x}) & \text{for acoustic spaces} \\ \left\langle v_i^2(\mathbf{x})\right\rangle = \sum\limits_{k=1}^{N_f}\left\langle v_{i,k}^2(f_k,\mathbf{x})\right\rangle = \dfrac{1}{m_i}\sum\limits_{k=1}^{N_f} E_{i,k}(f_k,\mathbf{x}) & \text{for structures} \end{cases} \quad (5)$$

where: N_f is the number of response frequency bands.

STATISTICAL DISTRIBUTION OF SEA RESPONSES

The distributions of the SEA model responses considered here originate from randomness of design parameters with arbitrary distributions. The deviations of SEA responses due to design parameters are approximated by a Taylor series expansion (Radcliffe and Huang, 1993). Randomly distributed design parameters yield SEA response variations through SEA equations characterized by their distributions. The proof of Gaussian distributions proceeds with a statement of assumptions, expansion of SEA response as a Taylor series, application of the Central Limit Theorem, and confirmation that the assumptions are valid.

1: Assumptions. (1): SEA response derivatives with respect to SEA parameters and SEA parameter derivatives to design parameters are bounded above. Assumption (2): Third order absolute moments of all design parameters are bounded above. Assumption (3): no single design variable dominates.

2: Taylor Expansion of SEA Responses. The SEA responses are non-linearly and implicitly related to design parameters. The derivation of probability distributions for SEA responses requires the SEA responses be linearly and explicitly expressed in terms of design parameters by Taylor Expansions.

$$\begin{aligned} E_{i,k} &\approx \bar{E}_{i,k} + \sum_{m=1}^{N_x}\frac{\partial E_{i,k}}{\partial x_m}\bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}}\\ \mathbf{P}=\bar{\mathbf{P}}}}(x_m-\bar{x}_m) \\ &\approx \bar{E}_{i,k} + \sum_{m=1}^{N_x}(y_m-\bar{y}_m) \end{aligned} \quad (6)$$

where: $\bar{E}_{i,k}$ is the expected value of SEA responses,
N_x is the number of design parameters,
$\dfrac{\partial E_{i,k}}{\partial x_m}$ is SEA response derivative for design parameters,

$$y_m = \frac{\partial E_{i,k}}{\partial x_m}\bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}}\\ \mathbf{P}=\bar{\mathbf{P}}}} x_m \text{ and } \bar{y}_m = \frac{\partial E_{i,k}}{\partial x_m}\bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}}\\ \mathbf{P}=\bar{\mathbf{P}}}} \bar{x}_m$$

The partial derivatives above are evaluated at the nominal design parameters $\mathbf{x}=\bar{\mathbf{x}}$ and nominal power inputs $\mathbf{P}=\bar{\mathbf{P}}$ (Huang and Radcliffe, 1993).

Step 3: The Central Limit Theorem (CLT) and Lyapunov Condition. CLT can be invoked to derive the distributions of SEA energy responses which are sums consisting of a number of independent random variables with arbitrary distributions (6). The CLT states (Cramér, pg. 215, 1963): let $y_1, y_2, \cdots, y_n$ be independent variables with arbitrary distributions, finite expected values and variances, $\sigma^2(y_i)$. If a Lyapunov Condition:

$$\omega_n = \sqrt[3]{\sum_{m=1}^{n} M_3\left(|y_m-\bar{y}_m|\right)} \Bigg/ \sqrt{\sum_{m=1}^{n}\sigma^2(y_m)} \to 0 \quad as \quad n\to\infty \quad (7)$$

where: $M_3\left(|y_m-\bar{y}_m|\right)$ is the third order absolute moment of y_m about mean $\bar{y}_m$ and
n is the number of variables,

is satisfied, then the random variable, $z = \sum\limits_{i=1}^{n} y_i \Big/ \sqrt{\sum\limits_{i=1}^{n}\sigma^2(y_i)}$ tends asymptotically to have a Gaussian distribution.

The third order absolute moment $M_3\left(|y_m-\bar{y}_m|\right)$ and variance $\sigma^2(y_m)$ of intermediate variable y_m about its mean $\bar{y}_m$ in (6) are needed to check the Lyapunov condition for SEA responses. The third order absolute central moment of y_m for SEA responses can be expressed in terms of the third order absolute moments of x_m about its mean $\bar{x}_m$ using the linearity property of the moments.

$$M_3\left(|y_m-\bar{y}_m|\right) = \left[\left[\frac{\partial E_{i,k}}{\partial x_m}\bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}}\\ \mathbf{P}=\bar{\mathbf{P}}}}\right]\right]^3 M_3\left(|x_m-\bar{x}_m|\right) \quad (8)$$

The variance $\sigma^2(y_m)$ of y_m can be expressed in terms of variance $\sigma^2(x_m)$ of x_m.

$$\sigma^2(y_m) = \left[\left[\frac{\partial E_{i,k}}{\partial x_m}\bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}}\\ \mathbf{P}=\bar{\mathbf{P}}}}\right]\right]^2 \sigma^2(x_m) \quad (9)$$

The Lyapunov Condition for SEA energy responses can be obtained by substituting (8) and (9) into (7).

$$\omega_{N_x} = \sqrt[3]{\sum_{m=1}^{N_x}\left[\left[\frac{\partial E_{i,k}}{\partial x_m}\bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}}\\ \mathbf{P}=\bar{\mathbf{P}}}}\right]\right]^3 M_3\left(|x_m-\bar{x}_m|\right)} \Bigg/ \sqrt{\sum_{m=1}^{N_x}\left[\left[\frac{\partial E_{i,k}}{\partial x_m}\bigg|_{\substack{\mathbf{x}=\bar{\mathbf{x}}\\ \mathbf{P}=\bar{\mathbf{P}}}}\right]\right]^2 \sigma^2(x_m)} \quad (10)$$

where N_x = number of random SEA model design parameters.

If the upper bound of ω_{N_x} in (10) approaches zero as N_x goes to infinity, (10) will satisfy the Lyapunov Condition (7) for SEA energy responses. The Lyapunov Condition (10) is determined by three quantities, SEA response derivatives, $\partial E_{i,k}/\partial x_m$, design parameter third order absolute central moments $M_3(|x_m-\bar{x}_m|)$ and variances, $\sigma^2(x_m)$. The absolute values of SEA response derivatives under assumption (1) are bounded above and therefore have a maximal value.

$$\max\left\{\left|\frac{\partial E_{i,k}}{\partial \mathbf{x}}\right|\right\}=\max\left\{\left|\frac{\partial E_{i,k}}{\partial x_m}\right|_{\substack{\mathbf{x}=\bar{\mathbf{x}}\\ \mathbf{P}=\bar{\mathbf{P}}}},\ m=1,2,\cdots,N_x\right\} \tag{11}$$

The absolute values of SEA response derivatives under assumption (1) are bounded below away from zero and have a minimal value by ignoring the trivial case of the zero-SEA parameter derivatives.

$$0<\min\left\{\left|\frac{\partial E_{i,k}}{\partial \mathbf{x}}\right|\right\}=\min\left\{\left|\frac{\partial E_{i,k}}{\partial x_m}\right|_{\substack{\mathbf{x}=\bar{\mathbf{x}}\\ \mathbf{P}=\bar{\mathbf{P}}}},\ m=1,2,\cdots,N_x\right\} \tag{12}$$

An upper bound for the Lyapunov Condition (10) can be obtained by replacing the derivatives in the numerator with the maximal value and replacing the derivatives in the denominator with the minimal value.

$$\begin{aligned}\omega_{N_x}&\le \max_m\left\{\frac{\partial E_{i,k}}{\partial \mathbf{x}}\right\}\sqrt[3]{\sum_{m=1}^{N_x}M_3(|x_m-\bar{x}_m|)}\Big/\min\left\{\frac{\partial E_{i,k}}{\partial \mathbf{x}}\right\}\sqrt{\sum_{m=1}^{N_x}\sigma^2(x_m)}\\&=C_{mm}\sqrt[3]{\sum_{m=1}^{N_x}M_3(|x_m-\bar{x}_m|)}\Big/\sqrt{\sum_{m=1}^{N_x}\sigma^2(x_m)}\end{aligned} \tag{13}$$

where $C_{mm}=\max\left\{\dfrac{\partial E_{i,k}}{\partial \mathbf{x}}\right\}\Big/\min\left\{\dfrac{\partial E_{i,k}}{\partial \mathbf{x}}\right\}$

The bounds of the variances and moments can be used to further simplify the upper bound of ω_{N_x} (13). The third order absolute moments of design parameters, $M_3(|x_m-\bar{x}_m|)$, are bounded above under assumption (2). Therefore the third absolute central moments have a maximal value. The zero-variance of a variable means that the variable is a constant, the zero-variance variables can be excluded from the moment and variance calculation. Hence the variances are bounded below away from zero and have a non-zero minimal value. The upper bound for the Lyapunov Condition (13) can be simplified by replacing the summation for the third absolute central moments in the numerator with the product of the moment maximal value by the number of the design parameters and the summation for the variances in the denominator with the product of the non-zero variance minimal value by the number of the design parameters.

$$\begin{aligned}\omega_{N_x}&\le C_{mm}\sqrt[3]{\max_m\{M_3(|x_m-\bar{x}_m|)\}N_x}\Big/\sqrt{\min_m\{\sigma^2(x_m)\}N_x}\\&=D_{mm}/\sqrt[6]{N_x}\end{aligned} \tag{14}$$

where: $D_{mm}=C_{mm}\sqrt[3]{\max_m\{M_3(|x_m-\bar{x}_m|)\}}\Big/\sqrt{\min_m\{\sigma^2(x_m)\}}$

The Lyapunov Condition for SEA energy response in (14) has a constant D_{mm} in the numerator and N_x in the denominator. Letting N_x go to infinity, the constant term in the numerator remains unchanged and the denominator, $\sqrt[6]{N_x}$, goes to infinity, so $\omega_{N_x}\to 0$. The Lyapunov condition is satisfied which guarantees that probability distributions of SEA responses, $E_{i,k}$, tend asymptotically to be Gaussian with a mean of $\bar{E}_{i,k}$ and a variance,

$$\sigma^2(E_{i,k})=\sum_{m=1}^{N_x}\left[\left.\frac{\partial E_{i,k}}{\partial x_m}\right|_{\substack{\mathbf{x}=\bar{\mathbf{x}}\\ \mathbf{P}=\bar{\mathbf{P}}}}\right]^2\sigma^2(x_m).$$

Step 4: Assumption Check on SEA Response Derivatives. The SEA response derivatives with respect to design parameters can be obtained by taking derivative on both sides of (3):

$$\frac{\partial E_{i,k}}{\partial x_m}=\frac{\partial e_{i,k}}{\partial x_m}n_i+\frac{\partial n_i}{\partial x_m}e_{i,k} \tag{15}$$

The SEA modal response derivatives with respect to SEA parameters, $\partial e_{i,k}/\partial x_m$, in the first term in (15) are the product of three terms: $\mathbf{N}^{-1}$, $\partial\mathbf{N}/\partial x_m$ ($x_m=n_i,\eta_i$ *and* η_{ij}) and $\mathbf{e}$ (Radcliffe and Huang, 1993). For an SEA model, $\mathbf{N}$ is positive definite, diagonally dominant and $\mathbf{N}^{-1}$ exists and is diagonally dominant (Hodges, et al, 1987, Huang and Radcliffe, 1993). All entries in $\mathbf{N}$ are linear combinations of modal densities, internal loss factors and coupling loss factors. These entries are finite for finite frequencies. The entries in $\mathbf{N}$ are bounded by their diagonal entries and so are the entries in $\mathbf{N}^{-1}$. The entries in derivatives are modal densities, internal loss factors and coupling loss factors with more zeros (Huang and Radcliffe, 1993). So the entries in $\partial\mathbf{N}/\partial x_m$ are bounded for finite frequencies. The modal energy response, $\mathbf{e}$, is associated linearly to power inputs which drive the systems. The modal energy response, $\mathbf{e}$, is bounded over finite frequency bands for bounded power inputs. Therefore the SEA response derivatives with respect to SEA parameters in (15) are bounded.

Assumption (2) sets an upper bound for the third order absolute moments of design parameters. The larger the upper bound the slower the Lyapunov Condition goes to zeros. This feature has significance in engineering applications. There is never infinite number of design parameters in any SEA models in engineering. Gaussian distribution derived here is only an approximation for SEA responses in engineering applications. The small third order absolute moments mean small deviations among design parameters. The small deviations from nominal design parameters will lead to good approximations to Gaussian distributions. In other words, for small third order absolute moment upper bound, only a small number of random design parameters are needed to achieve good approximations to Gaussian distribution. Assumption (3) guaranteed that no single random design parameter, x_m, in (6) is statistically dominant. Otherwise, the SEA response distribution will follow the distribution of the dominant random variable in (6).

MONTE CARLO ANALYSIS

Monte Carlo analysis of a simple model (Figure 1 and 2) is used here to test the theoretical statistical distribution results. The SEA model has 18 design parameters and 1 power input. Power inputs are set inversely proportional to band center frequency: $P_i=f_i^{-1.5}\times 10^{-4}$ Watts. Formulas used to calculate SEA parameters are from Lyon (1975). Only geometric parameters (Table A1, Appendix), such as lengths and thicknesses are assumed to have Gaussian distributions with design values as means and 5% of the design values as their standard deviations. All other parameters, such as mass densities, speeds of sound, reverberation times, damping ratio, and input powers, are assumed to have uniform distributions in intervals of ±10% from their nominal mean values.

The Monte Carlo analysis proceeds by generating 2,000 independent, randomly parameterized models with randomly distributed parameters from the distributions described above. For each model, SEA responses are computed in 20 one-third octave bands with center frequencies ranging from 125 Hz. to 10,000 Hz. For each of the 2000 independent sets of one-third octave, model responses, the 20 one-third octave responses are summed to compute 2000 total mean-squared acoustic pressures and structural velocities. The result is a 2000 member, independent, randomly distributed sample of the statistical population of possible SEA responses. This 2000 member sample is used to verify the hypothetical Gaussian distributions.

Histograms for one-third octave responses at 1,000 Hz (Fig. 3) and total mean-squared responses (Fig. 4) are shown.

The bars represent the histograms from Monte Carlo test runs and the solid lines represent a Gaussian distribution computed from the means and standard deviations of the Monte Carlo samples. The histograms show that the distributions of both one-third octave and total mean-squared acoustic sound pressure and structural velocity are well approximated by Gaussian distributions. Previous work has shown that the means and standard deviations computed from the Monte Carlo results are also well predicted by variance analysis (Radcliffe and Huang, 1993).

χ^2 HYPOTHESIS TEST

A χ^2 (Chi-Squared) hypothesis test is used here to analytically examine the agreement between the hypothetical Gaussian distribution approximations for infinite numbers of model parameters and the finite order distributions from the Monte Carlo test of the SEA model results. The objective of the Chi-Squared test is to determine if the deviations of sample distributions from hypothetical distributions are small enough to accept the hypothesis. The Chi-Squared deviation measure (16) is quantified by the normalized ms difference between cumulative frequency from the sample and from the hypothesis.

$$\chi^2 = \sum_{i=1}^{R}\left(F_i - N_{mc}P(A_i)\right)^2 \Big/ N_{mc}P(A_i) \qquad (16)$$

where N_{mc} is the number of tests
A_i is a non-overlapping data interval
$P(A_i)$ is the hypothetical data probability in interval A_i
F_i is actual Monte Carlo test data frequency in A_i, and
R is the number of intervals, A_i

The occurrence frequency, F_i, and the data interval, A_i, of the SEA response statistics are determined from the Monte Carlo test histograms (Fig. 3-4). The median x_i of the interval, A_i, and the frequency, F_i, are used to estimate the mean, $\bar{x}_i$, and standard deviations, $\sigma(x_i)$, of the hypothetical Gaussian distribution by the maximum likelihood method. The estimated expected mean and standard deviation (17) are then used to evaluate the Gaussian distribution and to compute the probability $P(A_i)$.

$$\left\{\begin{aligned} \bar{\mu} &= \frac{1}{2{,}000}\sum_{i=1}^{15} F_i x_i = 4.909\times10^{-3}\ (\mathrm{P_a^2}) \\ \bar{s} &= \sqrt{\frac{1}{2{,}000}\left(\sum_{i=1}^{15}\left(F_i x_i^2\right) - \bar{\mu}^2\right)} = 4.257\times10^{-4} \end{aligned}\right. \qquad (17)$$

Chi-Squared test results for total mean-squared sound pressures and plate velocity are listed in Tables 1-3. For each Chi-Squared test, the amplitude predictions from $N_{mc} = 2{,}000$ Monte Carlo analyses are divided into 15 amplitude intervals similar to those of the histograms in Fig. 3-4. The computation of histogram intervals, as well as estimates of mean and standard deviation reduces the statistical degrees-of-freedom from the fifteen (15) histogram levels to twelve (12) used in the initial estimates of Chi-Squared from the test results.

The Chi-Squared probability, $P(\chi_G^2 > \chi^2)$, is a function of both Chi-Squared value, χ^2, and degrees-of-freedom, ν, in the distribution (18).

$$P = 1 - \left\{ \int_0^{\chi^2} e^{-\frac{t}{2}}\, t^{\frac{\nu}{2}-1}\, dt \Big/ \int_0^{\infty} e^{-t}\, t^{\frac{\nu}{2}-1}\, dt \right\} \qquad (18)$$

The Chi-Squared probability, $P(\chi_G^2 > \chi^2)$, represents the probability that a hypothetical Gaussian distribution would have a Chi-Squared value greater than, or, equal to the estimated Chi-Squared value. Values of P above 0.95 are recognized to indicate unusually large, suspiciously non-random, correlation with the hypothetical Gaussian distribution. Typically, values of P below 0.001 indicate a distribution that has Chi-Squared values much above those expected from the hypothetical Gaussian distribution and the Hypothesis is rejected. Values of P below 0.95 and above P=0.001, 0.01, or 0.05 are considered to be Gaussian at a "probably significant", "significant" and "highly significant" level respectively(Beckwith &Marangoni, 1990).

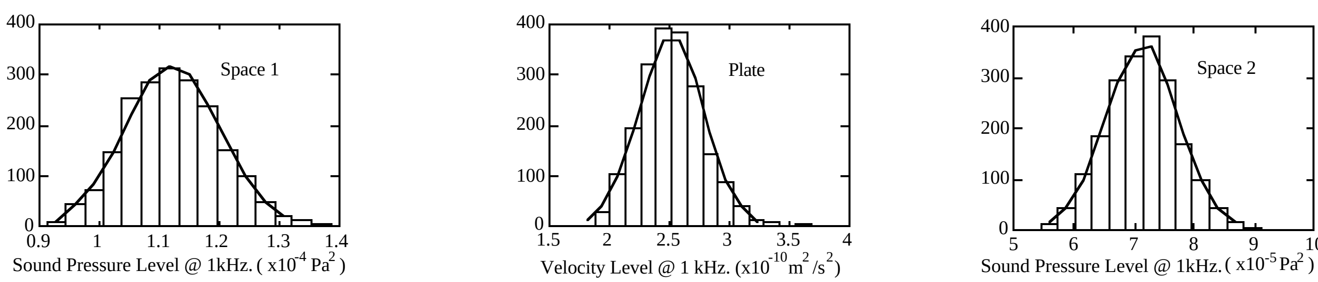

Figure 3: Monte Carlo Test Histogram @ 1 kHz.

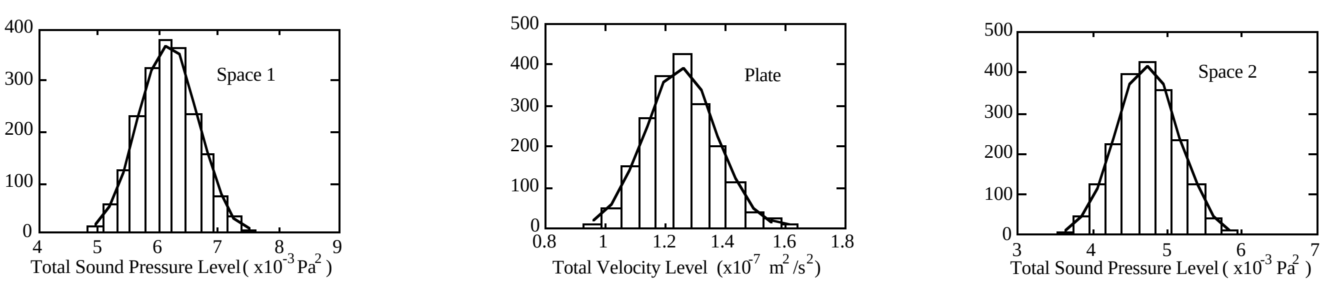

Figure 4: Monte Carlo Test Histogram for 125 to 10 kHz.

Table 1: χ^2 Test for Mean-Squared Sound Pressure in Space 1. The Probability, P, the Distribution is Gaussian is Indicated with the Degrees-of-Freedom of the Chi-Squared Distribution Shown in Parentheses.

Frequency Band	$\chi^2(12)$	$\chi^2_\alpha\ (d.o.f)$	P
500 Hz.	7.297	7.584 (10)	0.84
1,000 Hz.	9.894	12.55 (9)	0.63
5,000 Hz.	9.268	9.342 (9)	0.68
125 Hz. - 10k Hz.	6.784	8.343 (8)	0.87

Table 2: χ^2 Test for Plate Mean-Squared Velocity

Frequency Band	$\chi^2(12)$	$\chi^2_\alpha\ (d.o.f)$	P
500 Hz.	16.42	16.81 (5)	0.17
1,000 Hz.	17.45	19.68 (8)	0.13
5,000 Hz.	18.89	22.43 (5)	0.09
125 Hz. - 10k Hz.	18.22	22.43 (5)	0.11

Table 3: χ^2 Test For Sound Pressure in Space 2

Frequency Band	$\chi^2(12)$	$\chi^2_\alpha\ (d.o.f)$	P
500 Hz.	5.709	8.383 (6)	0.93
1,000 Hz.	7.855	8.343 (8)	0.80
5,000 Hz.	8.493	10.66 (8)	0.75
125 Hz. - 10k Hz.	5.717	7.344 (7)	0.93

The highest probability value, P, from both Chi-Squared value estimates is used as a best estimate of the probability, P. The Probability estimates clearly demonstrate that the Monte Carlo analysis results fit Gaussian distributions for both sound pressure and plate velocities. The results for sound pressure have a higher probability, P, value than those for plate velocity. This result is a likely result of the relatively larger modal densities and larger numbers of modes in the acoustic volumes relative to those of the plate model. The results also show that the results for acoustic response are even more likely to appear Gaussian than the results for response of the plate.

CUMULATIVE SEA RESPONSE DISTRIBUTIONS AND CONFIDENCE LEVELS

A cumulative Gaussian probability graph provides a third method for measuring the accuracy of the Gaussian distribution approximation. Points in this graph represent the cumulative frequency of occurrence, or percentile, associated with results that have amplitudes less or equal to the deviation coordinate. The Gaussian (normal) probability graph used here is scaled such that all cumulative percent frequency points from a Gaussian distribution lie in a straight line (Volk, 1969). Statistical cumulative distribution percentiles of Monte Carlo data for both narrow band response at 1000 Hz. (Fig. 5) and wide band response from 125 Hz. to 10 kHz. (Fig. 6), are plotted against response amplitude. For comparison, straight lines representing a true Gaussian distributions are shown. In both the 1,000 Hz. and wide band results, the response amplitude percentiles from the Monte Carlo analysis results are within 1% of Gaussian distribution values for percentiles ranging from 2% to 98%. As expected, percentiles near the edges of the distributions with small numbers of samples are farthest from Gaussian percentiles. The plate velocity results have the greatest deviation from Gaussian distributions, however, even the plate velocity results are well approximated by Gaussian distributions from 2% to 98% levels. The acoustic pressure result percentiles are remarkably well represented by the Gaussian approximation and are within 0.1% of Gaussian percentiles to the 99.8% percentile level.

Design Confidence level are simply extensions of these cumulative distributions. Models with a large number of independent parameters will approach the theoretical Gaussian approximation. The example used here has only 18 independent model parameters and does not yet approach a Gaussian distribution for all predicted amplitudes. Even in this small model case, the cumulative distribution results allow the Gaussian approximation to be used acceptably for model response percentile estimates to 98% for velocity and 99.8% for acoustic pressures. For example, 98% of all example response levels are less than 2.8 standard deviations above the design mean levels. This finite number of parameter example indicates that the Gaussian approximation should be a reasonable estimate of true design distributions for this example and larger engineering SEA models.

CONCLUSIONS

In this paper, the distributions of predicted sound and vibration responses from Statistical Energy Analysis (SEA) are proven to be Gaussian for an infinite number of model parameters regardless of distributions of those design parameters. This Gaussian distribution result provides a useful approximation for vibro-acoustic SEA models with a finite number of model parameters. In practice, the assumption of Gaussian distributions combined with previously derived analytical expressions (Radcliffe and Huang, 1993), allow easily computed estimates for the variations expected in SEA response results given the variances in model parameters. Gaussian distribution results allow the computation of confidence levels for worst case estimates of sound pressure and vibration levels in engineering vibro-acoustic models. These worst case estimates are critical to

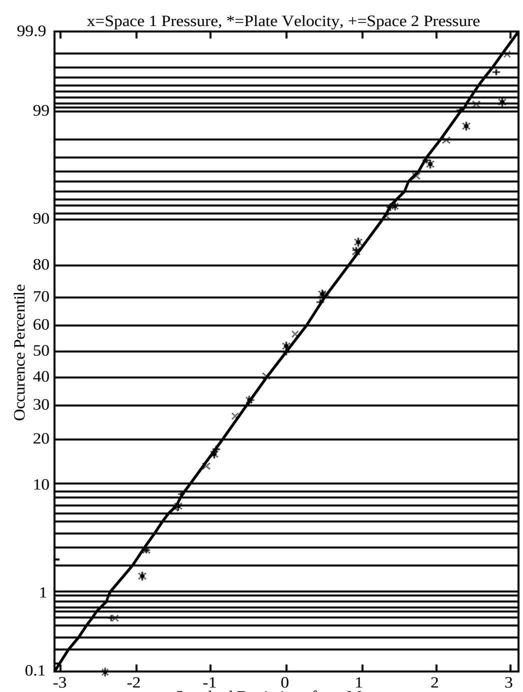

Figure 5: 1.0 kHz Sound Pressure and Structural Velocity Cumulative Distributions. Solid Line is a Gaussian Distribution. Points Are Monte Carlo Results

the use of Statistical Energy Analysis as a modern design tool for development of high quality designs where vibro-acoustic response is an important specification. One important application area is automotive vehicles because customer perception of automobile quality is tightly coupled to low operating noise and vibration.

The Gaussian response theoretical result has been verified with a Monte Carlo analysis of a three element SEA model with eighteen (18) model parameters. In this analysis, 2,000 SEA models were generated for each of 20 one-third octave frequency bands. χ^2 statistical tests conducted to test the statistical population of the SEA responses indicate that the Monte Carlo results are well approximated by a Gaussian distribution even when the SEA model has only eighteen design parameters. Percentile graphs of model acoustic pressure and plate vibration levels, demonstrate that the Gaussian approximation is accurate to at least the 98% percentile. Gaussian distribution of SEA responses can, therefore, be applied to evaluate confidence levels of the existing acoustic designs and develop specifications for new designs. With these features and applications, the Gaussian distribution approximation for SEA responses provides a powerful design tool for quality acoustic design of aerospace, ship and automotive vehicle systems.

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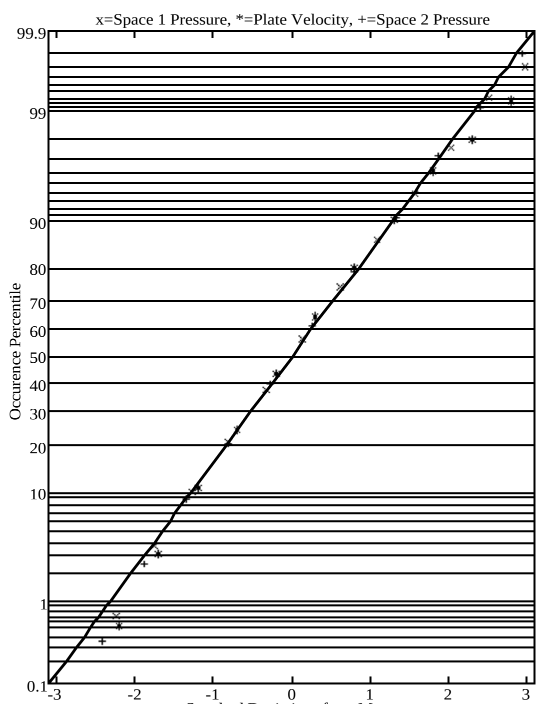

Figure 6: Total Sound Pressure and Structural Velocity Cumulative Distribution. Solid Line is Gaussian Distribution. Points Are Monte Carlo Results

APPENDIX

Table A1: Geometric and Material Properties Used in the Example to Form 18 Independent Model Parameters, each with the Distribution Type Indicated

Parameter	n	Mean, μ	Distribution
Plate Longitudinal Wave Speed (m/s)	1	5181.0	Uniform, ±0.10μ
Speed of Sound in Air (m/s)	2	344	Uniform, ±0.10μ
Plate Thickness (m)	1	0.00075	Gaussian, s=0.05μ
Length of the Cube (m)	6	0.5	Gaussian, s=0.05μ
Air Density (kg/m^3)	2	1.244	Uniform, ±0.10μ
Aluminum Density (kg/m^3)	1	2700	Uniform, ±0.10μ
Speed of Sound in Aluminum (m/s)	1	6400	Uniform, ±0.10μ
Space 1 Reverberation Time (sec)	1	1.5	Uniform, ±0.10μ
Space 2 Reverberation Time (sec)	1	1.2	Uniform, ±0.10μ
Flanking Coupling Loss Factor	1	0.0001	Uniform, ±0.10μ
Damping Ratio in Plate	1	0.001	Uniform, ±0.10μ