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A Nyquist Stability Criterion for Distributed Parameter Systems

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Abstract—**A Nyquist graphical stability criterion is developed for distributed parameter, possibly unstable, single-loop systems. Practical conditions are presented for existence, uniqueness, causality, and asymptotic or exponential stability of the closed-loop impulse response. The hypotheses are given for the transfer function only and do not require any knowledge of its time domain impulse response.**

INTRODUCTION

The Nyquist graphical stability criterion was extended in [1] and [2] to systems with nonrational, possibly unstable, transfer functions. In [1] and [2], it is assumed that the impulse response can be separated into two parts: a part arising from a finite number of real or complex-conjugate pairs of unstable poles and a second part arising from a causal function consisting of the sum of an absolutely integrable function and absolutely summable delays. The hypotheses for the graphical stability criterion presented below are given for the transfer function only and do not require any knowledge of its time domain impulse response form.

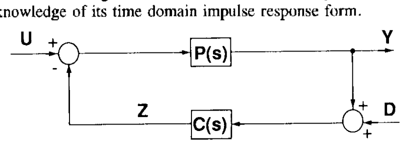

Fig. 1. The control system.

RESULTS

Consider the control system shown in Fig. 1. The system is described by the four transfer functions: $Y(s)/U(s) \triangleq H(s) = P(s)/[1 + Q(s)]$, $Y(s)/D(s) = -Q(s)/[1 + Q(s)]$, $Z(s)/U(s) = -Y(s)/D(s)$, and $Z(s)/D(s) = C(s)/[1 + Q(s)]$ where $P(s)$ is possible nonrational, $Q(s) = P(s)C(s)$, and where $C(s)$ is a proper rational transfer function. It is often the case that $P(s)$, arising in a distributed parameter control system, satisfies, for some nonnegative real constant σ_0, the following properties:

A1: $P(s)$ is meromorphic [3] in the finite right half-plane Re $(s) \geq -\sigma_0$.

A2: $P(s)$ and its first two derivatives are absolutely integrable (L^1) on the vertical line Re $(s) = -\sigma_0$ outside some bounded subinterval of the line.

A3: $P(s)$ and its first two derivatives vanish as $|s| \rightarrow \infty$ on the closed right half-plane Re $(s) \geq -\sigma_0$.

A4: There are no zero/pole cancellations in $P(s)C(s)$ in the closed right half-plane Re $(s) \geq -\sigma_0$.

The results in this section are given for $H(s)$ only. Results for the other three transfer functions follow from the Theorem and are given afterwards.

Remark 1: $P(s)$ can include unstable elements. Delays are covered as long as the above properties hold.

Remark 2: $H(s)$ is meromorphic in Re $(s) \geq -\sigma_0$, since the sum, product, and quotient of meromorphic functions are again meromorphic.

Remark 3: $H(s)$ vanishes as $|s| \rightarrow \infty$ on Re $(s) \geq -\sigma_0$ because its denominator $1 + Q(s)$ is essentially 1 for large $|s|$. In fact, $H(s)$ is dominated in magnitude by $kP(s)$ on Re $(s) \geq -\sigma_0$ for large $|s|$ and some constant k. Likewise, $H'(s)$ and $H''(s)$ are dominated by certain derivatives of $P(s)$ and thus vanish as $|s| \rightarrow \infty$ on Re $(s) \geq -\sigma_0$.

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Remark 4: Because $H(s)$ is meromorphic and strictly proper on Re $(s) \geq -\sigma_0$, then $H(s)$ can have at most a finite number of poles in Re $(s) \geq -\sigma_0$. As a result, the contour of the Nyquist graphical test can be finite.

For a stability theorem to make any sense at all, then existence, uniqueness, and causality of the closed-loop impulse response $h(t)$ defined by the inversion formula

$$h(t) \triangleq \int_{-\sigma_0-j\infty}^{-\sigma_0+j\infty} H(s)e^{st}\,ds/2\pi j = \int_{-\infty}^{\infty} H(-\sigma_0+j\omega)e^{(-\sigma_0+j\omega)t}\,d\omega/2\pi \quad (1)$$

must be demonstrated. We begin with a lemma showing that properties A1–A4 plus a Nyquist-like criterion imply that $H(s)$ is analytic on Re $(s) \geq -\sigma_0$. We then proceed to show existence, uniqueness, causality, and stability.

Lemma: Suppose that $P(s)$ of a linear time-invariant system, shown in Fig. 1, satisfies properties A1–A4. If the Nyquist plot of $Q(s)$ encircles the point $(-1, 0)$ p_0 times counterclockwise, where p_0 denotes the number of poles of $Q(s)$ in Re $(s) > -\sigma_0$, then $H(s)$ is analytic on Re $(s) \geq -\sigma_0$ (for a Nyquist plot definition see, for example, [4]).

Proof: By the Argument Principle [3], $1 + Q(s)$ has no zeros on Re $(s) \geq -\sigma_0$, and since $H(s)$ is meromorphic on Re $(s) \geq -\sigma_0$, it follows that $H(s)$ is analytic on Re $(s) \geq -\sigma_0$.

Theorem: If the hypotheses on $P(s)$ given in the Lemma are satisfied, then the impulse response $h(t)$ exists, is unique, causal, and asymptotically stable when $\sigma_0 = 0$ and exponentially stable when $\sigma_0 > 0$.

Proof:

Existence: Because $H(s)$ is continuous and is dominated by $P(s)$ on Re $(s) \geq -\sigma_0$, and since $P(s)$ is, by A2, eventually L^1 on Re $(s) = -\sigma_0$, then

$$\int_{-\infty}^{\infty} |H(-\sigma_0+j\omega)|\,d\omega \leq \int_{-\Omega}^{\Omega} |H(-\sigma_0+j\omega)|\,d\omega + k\int_{-\infty}^{-\Omega} |P(-\sigma_0+j\omega)|\,d\omega + k\int_{\Omega}^{\infty} |P(-\sigma_0+j\omega)|\,d\omega < \infty \quad (2)$$

for some large positive Ω and some positive constant k. Hence, $H(s)$ is L^1 over the entire vertical line Re $(s) = -\sigma_0$. Thus, the integral (1) converges absolutely.

Causality: Because $H(s)$ is analytic on Re $(s) \geq -\sigma_0$, by the Cauchy Theorem [3] the integral (1) can be separated into three contour integrals as follows:

$$h(t) = \int_{\Gamma} H(s)e^{st}\,ds/2\pi j + e^{-\sigma_0 t}\int_{\Omega}^{\infty} H(-\sigma_0+j\omega)e^{j\omega t}\,d\omega/2\pi + e^{-\sigma_0 t}\int_{-\infty}^{-\Omega} H(-\sigma_0+j\omega)e^{j\omega t}\,d\omega/2\pi \quad (3)$$

where Ω is a large positive number and Γ denotes the semicircle $s = -\sigma_0 + \Omega e^{j\theta}$, $-\pi/2 \leq \theta \leq \pi/2$. Because $H(s)$ vanishes as $|s| \rightarrow \infty$ on Re $(s) \geq -\sigma_0$, the Jordan lemma [5] guarantees that the integral along Γ approaches zero as $\Omega \rightarrow \infty$ for $t < 0$. Because $H(-\sigma_0 + j\omega)$ is L^1, the last two integrals can be made arbitrarily small for sufficiently large Ω. Therefore, $h(t) = 0$ for $t < 0$.

An easier but nontraditional proof of causality can be obtained by employing a rectangular contour rather than the above semicircle employed in the Jordan lemma.

Remark 5: Because $H(s)$ is L^1, $h(t)$ is uniformly continuous by the Lebesgue bounded convergence theorem [6].

Stability: Because of Remark 3, the Lemma, and property A2, integrating the inversion formula (1) twice by parts yields, for $t > 0$

$$e^{\sigma_0 t}h(t) = -\frac{1}{t^2}\int_{-\infty}^{\infty} H''(-\sigma_0+j\omega)e^{j\omega t}\,d\omega/2\pi. \quad (4)$$

The product $e^{\sigma_0 t}h(t)$ is in $L^1[0, \infty)$ since the function on the right-hand side of (4) is of order $1/t^2$ at ∞. Thus, $h(t)$ is asymptotically stable when $\sigma_0 \geq 0$ and exponentially stable [2] when $\sigma_0 > 0$. By asymptotic stability we mean bounded-input bounded-output stability [1] plus $h(t) \rightarrow 0$ as $t \rightarrow \infty$.

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Uniqueness: Because both $H(s)$ and $h(t)$ are absolutely integrable and since $H(s)$ is differentiable, then by [7], $h(t)$ and $H(s)$ are a Laplace transform pair.

A second proof of the theorem can be obtained by appeal to the results in [1] and [2]. Splitting off the unstable singular part from $Q(s)$ to obtain a residual part analytic on Re $(s) \geq -\sigma_0$, and by employing at times conditionally convergent integral, it can be shown that the inverse Laplace transform of $Q(s)$ belongs to the class of systems considered in [1] and [2]. Thus, our hypotheses can be employed to decide whether or not a nonrational transfer function is covered by [1] and [2]. Our analysis also provides an independent proof for closed-loop impulse response stability.

The results for the transfer function $Q(s)/[1 + Q(s)]$ follow by similar arguments used in the proof of the theorem and are given in the Corollary 1.

Corollary 1: If the hypothesis given in the Theorem is satisfied, then the conclusions of the Theorem also hold for the impulse response of $Q(s)/[1 + Q(s)]$.

To consider the fourth transfer function $F(s) \triangleq C(s)/[1 + Q(s)]$, we must consider stability in the generalized sense [8].

Corollary 2: If the hypothesis given in the Theorem is satisfied and $C(s)$ is analytic on Re $(s) \geq -\sigma_0$, then $f(t)$ is stable in the generalized sense [8]. Moreover, if $C(s)$ is strictly proper, $f(t)$ is asymptotically stable when $\sigma_0 = 0$ and exponentially stable when $\sigma_0 > 0$.

Proof: In the time domain, the impulse response $f(t)$ is given by

$$f(t) = c(t) - c(t)*c(t)*h(t) \tag{5}$$

where '*' denotes generalized convolution [8]. Because $C(s)$ is proper, $f(t)$ is the difference of bounded-input bounded-output stable impulse responses. Moreover, if $C(s)$ is strictly proper, then $c(t)$ is exponentially stable. Therefore, because of (5), $f(t)$ and $h(t)$ share stability type. Note that $C(s)$ is not required to be stable in the Theorem and in Corollary 1.

Applications of the theorem can be found in [9] and [10].

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