Mathematical singularities associated with swelling of hyperelastic solids

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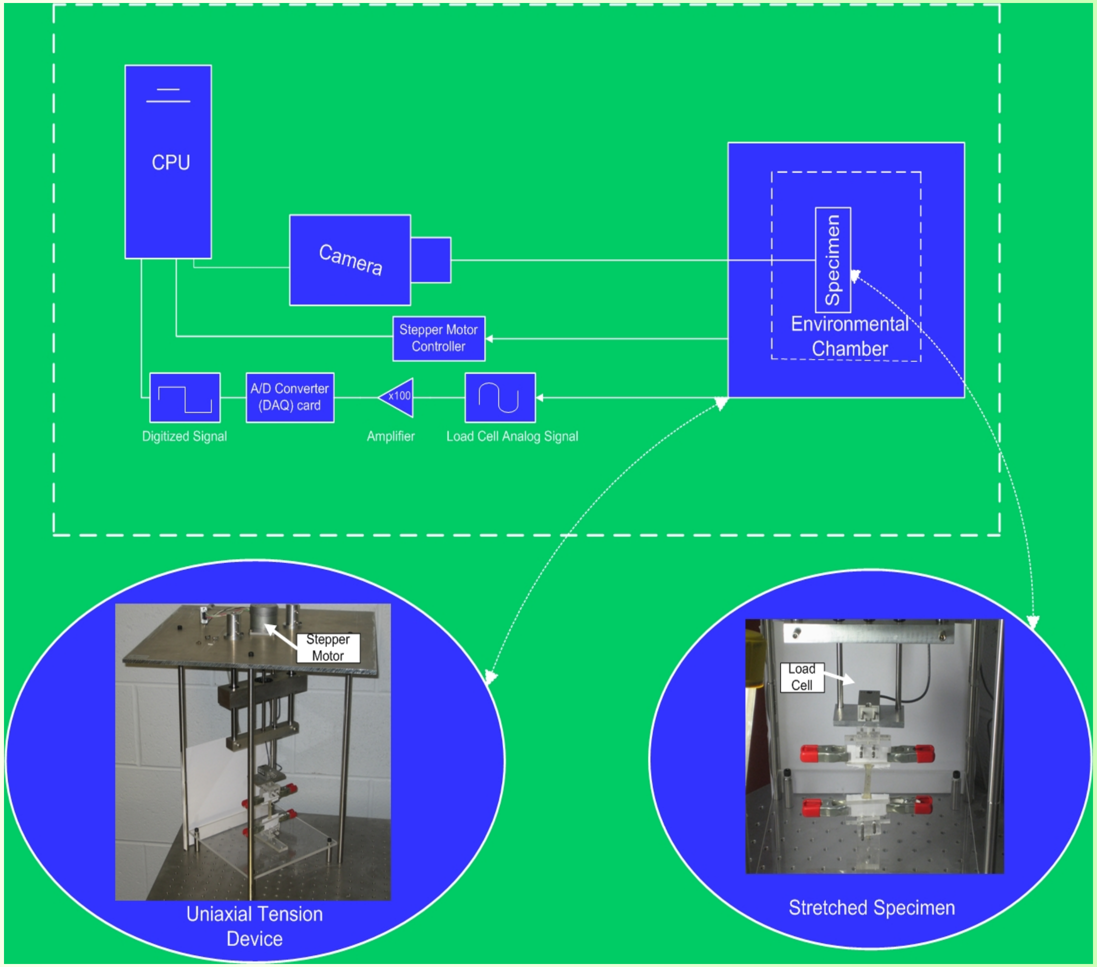


Swelling occurs in a variety of soft materials: rubbers and polymers, elastomer gels, soft materials.

source   www.math.utah.edu/~hsu/presentations/IGERT_pres by Prof. V. Hsu

"swelling" google hits: 17,300,00,
"swelling remedies" google hits: 1,500,000

A variety of mathematical treatments and traditions

- Poroelasticity with Darcy's law and Fick's law,
- Large deformation mechanics of mixtures theory (interacting media),
- Solid mechanics with energy densities that depend on a variable natural free volume.

Solid mechanics with energy densities that depend on a variable natural free volume is a simple generalization of the standard hyperelastic framework.

$\mathbf{y} = \mathbf{y}(\mathbf{X})$ maps reference configuration to deformed configuration,

with deformation gradient $$\mathbf{F} = \frac{\partial \mathbf{y}}{\partial \mathbf{X}}$$

volume change $J = \det \mathbf{F}$, if incompressible then $J = 1$

Elastic stored energy density $W = W(\mathbf{F}, \mathbf{X}, \underline{E})$ where $\underline{E}$ denotes a list of any constitutive parameters, e.g., stiffness moduli (convexity of wells).

Cauchy stress: $$\sigma = \frac{1}{J}\frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^T,$$ for compressible materials,

$$\sigma = \frac{1}{J}\frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^T - p\mathbf{I},$$ for incompressible materials.

The stress equations of equilibrium $div\boldsymbol{\sigma} = \mathbf{0}$ are then Euler-Lagrange equations for minimizing the total stored elastic energy.

Simple generalization to treat swelling is to introduce the natural free volume as a variable

v = natural free volume, dependent on temperature, chemical environment, stress, etc.

such that now $W=W(\mathbf{F}, \mathbf{X}, \underline{E}, v)$ and the usual minimization again gives $div\boldsymbol{\sigma} = \mathbf{0}$.

If the material is compressible after swelling, then $$\sigma = \frac{1}{J}\frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^T,$$

If the material is incompressible after swelling, then

$$\sigma = \frac{1}{v}\frac{\partial W}{\partial \mathbf{F}}\mathbf{F}^T - p\mathbf{I},$$ subject to $J = v.$

In addition to the usual "loading" due to tractions and body forces, there is the possibility of additional loading due to:

- confinement that gives rise to internal stress when swelling occurs,
- nonuniform swelling $v = v(\mathbf{X})$ in which case $div\boldsymbol{\sigma} = \mathbf{0}$ involves a "configurational force term" .

Dependence of *W* upon v

If $W(\mathbf{F}, \mathbf{X}, \underline{E})$ is a ``conventional'' hyperelastic energy density in that it registers an appropriate energy well structure for fixed free volume, then $W(v^{-1/3}\mathbf{F}, \mathbf{X}, \underline{E}(v))$:

- shifts energy wells in keeping with the notion of an energy minimal free volume by virtue of $v^{-1/3}\mathbf{F}$,
- allows for swelling dependent modulus changes by virtue of $\underline{E}(v)$.

Modulus parameters that grow slower than $O(v)$ give rise to modulus softening (as observed in gels) when mechanical response is linearized about the swollen state.

Example: conventional material (Shang-Cheng):

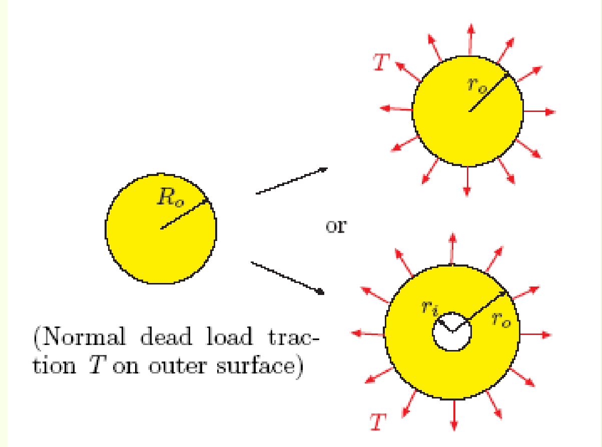
$$W = \frac{\mu}{2}\left(\lambda_1 + \lambda_2 + \lambda_3 + 3\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right) + 2\lambda_1\lambda_2\lambda_3 - 14\right)$$

with swelling:

$$W = \frac{\mu v^q}{2}\left(\frac{\lambda_1}{v^{1/3}} + \frac{\lambda_2}{v^{1/3}} + \frac{\lambda_3}{v^{1/3}} + 3\left(\frac{v^{1/3}}{\lambda_1} + \frac{v^{1/3}}{\lambda_2} + \frac{v^{1/3}}{\lambda_3}\right) + \frac{2}{v}\lambda_1\lambda_2\lambda_3 - 14\right)$$

$(0 < q < 1)$

Cavitation

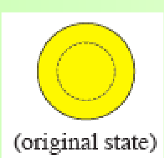


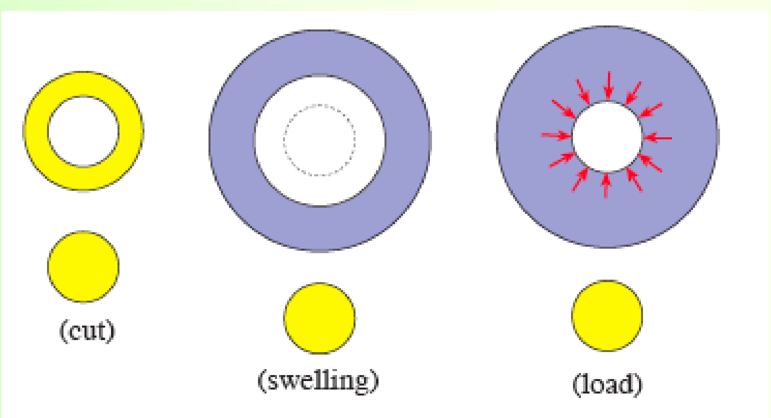
For swelling induced cavitation, consider a piecewise constant swelling field, where an outer layer swells while the inner core does not.

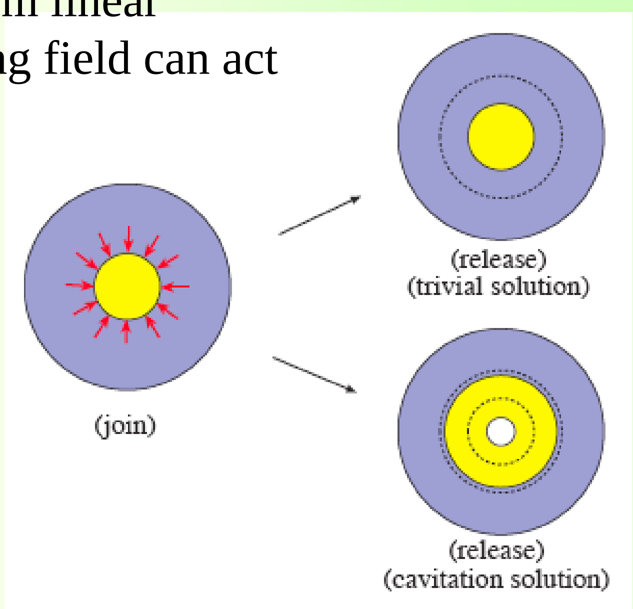
$$v = 1, \qquad 0 \leq R < R_m$$

$$v = \mathring{v} > 1, \qquad R_m < R \leq R_o$$

A “cut, swell, load, join, release” argument (familiar from linear eigenstrain/inclusion analysis) suggests how this swelling field can act as an effective loading device.







Spherical geometry: principal stretches are $\beta = \frac{\mathrm{d}r}{\mathrm{d}R}, \qquad \lambda = \frac{r}{R}$

with $\mathbf{F} = \beta \mathbf{e}_r \otimes \mathbf{e}_R + \lambda(\mathbf{e}_\theta \otimes \mathbf{e}_\Theta + \mathbf{e}_\phi \otimes \mathbf{e}_\Phi)$

$$\beta_i := \beta\Big|_{R\to 0+}, \qquad \beta_m^- := \beta\Big|_{R\to R_m^-}, \qquad \beta_m^+ := \beta\Big|_{R\to R_m^+}, \qquad \beta_o := \beta\Big|_{R=R_o},$$

$$\lambda_i := \lambda\Big|_{R\to 0+}, \qquad \lambda_m := \lambda\Big|_{R=R_m} \qquad \lambda_o := \lambda\Big|_{R=R_o}$$

The ODE associated with radial equilbrium admits a variety of solution alternatives:

- a conventionally smooth solution with no cavitation.
- a cavitation solution. (Preceded by a smooth solution with $\lambda_i \to \infty$)
- no smooth solutions capable of satisfying interfacial BC of λ continuity. (Preceded by a smooth solution with $\beta_m^+ \to \infty$)

Example: A swellable compressible surface layer over an incompressible core material that supports cavitation.

Layer $R_m \le R \le R_o$ is a compressible Shang-Cheng swelling material with q = 1/3,

Core $0 \le R \le R_m$ is an incompressible Varga material:

$$W_{core}(\lambda_1, \lambda_2, \lambda_3) = 2\mu_{core}(\lambda_1 + \lambda_2 + \lambda_3 - 3)$$

Smooth solutions exist provided the swelling is not too large

For given R_m and R_o let $\xi = R_m/R_o$ so that $0 \le \xi \le 1$. Then there exists $V_{max} = V_{max}(\xi)$ such that there is a unique smooth radially symmetric solution for $1 \le \mathring{v} < V_{max}(\xi)$ while there are no such smooth solutions for $\mathring{v} \ge V_{max}(\xi)$.

As $\mathring{v} \to V_{max}(\xi)$ one finds that $\beta_m^+ \to \infty$

Cavitation solutions are possible only if the core is sufficiently soft, whereupon such solutions exist if the swelling is sufficiently large.

$$\mu_{core} < \left(\frac{1}{4} + \frac{1}{2} V_{max}^{-2/3}\right)\mu \qquad \mathring{v} > V_{nuc} = V_{nuc}(\xi, \mu_{core}/\mu)$$

Thank You

Questions?