Solution of Additional Exercises for Chapter 4

1. **(1)** Try $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\dot{V}(x) = x_1(-x_1 + x_2^2) - x_2^2 = -x_1^2 - x_2^2 + x_1 x_2^2$$

In the neighborhood of the origin, the term $-(x_1^2 + x_2^2)$ dominates. Hence, the origin is asymptotically stable. Moreover

$$x_2(t) = e^{-t} x_{20} \ \Rightarrow x_1(t) = e^{-t} x_{10} + \int_0^t e^{-(t-s)} e^{-2s} \ ds \ x_{20}^2 = e^{-t} x_{10} + \left[e^{-t} - e^{-2t}\right] x_{20}^2$$

For all x_0, $x(t) \to 0$ as $t \to \infty$, which implies that the origin is globally asymptotically stable.
(2) Try $V(x) = ax_1^2 + bx_2^2$, $a > 0$, $b > 0$.

$$\dot{V}(x) = 2ax_1(x_1 - x_2)(x_1^2 + x_2^2 - 1) + 2bx_2(x_1 + x_2)(x_1^2 + x_2^2 - 1) = 2[ax_1(x_1 - x_2) + bx_2(x_1 + x_2)](x_1^2 + x_2^2 - 1)$$

Let $a = b$.

$$\dot{V}(x) = -2a(x_1^2 + x_2^2)[1 - (x_1^2 + x_2^2)]$$

For $x_1^2 + x_2^2 < 1$, $\dot{V}(x)$ is negative definite. Hence, the origin is asymptotically stable. It is not globally asymptotically stable since there other equilibrium points on the unit circle.
(3) Try $V(x) = \frac{1}{2}(ax_1^2 + bx_2^2)$, $a > 0$, $b > 0$.

$$\dot{V}(x) = ax_1(-x_1 + x_1^2 x_2) + bx_2(-x_2 + x_1) = -ax_1^2 + bx_1x_2 - bx_2^2 + ax_1^3 x_2 = -x^T Q x + ax_1^3 x_2$$

where $Q = \begin{bmatrix} a & -0.5b \\ -0.5b & b \end{bmatrix}$. The matrix Q is positive definite when $ab - b^2/4 > 0$. Choose $b = a = 1$. Near the origin, the quadratic term $-x^T Q x$ dominates the fourth-order term $x_1^3 x_2$. Thus, $\dot{V}(x)$ is negative definite and the origin is asymptotically stable. It is not globally asymptotically stable since there are other equilibrium points at $(1, 1)$ and $(-1, -1)$.

2. Consider $V(x) = x_1^2 + x_2^2$ as a Lyapunov function candidate.

$$\begin{aligned} \dot{V}(x) &= 2x_1^2(k^2 - x_1^2 - x_2^2) + 2x_1x_2(x_1^2 + x_2^2 + k^2) - 2x_1x_2(x_1^2 + x_2^2 + k^2) + 2x_2^2(k^2 - x_1^2 - x_2^2) \\ &= 2(x_1^2 + x_2^2)(k^2 - x_1^2 - x_2^2) \end{aligned}$$

(a)

$$k = 0 \ \Rightarrow \ \dot{V}(x) = -2(x_1^2 + x_2^2)^2$$

The origin is globally asymptotically stable.
(b)

$$k \neq 0 \ \Rightarrow \ \dot{V}(x) > 0, \quad \text{for } 0 < x_1^2 + x_2^2 < k^2$$

By Chetaev's theorem, the origin is unstable.

3. Try $g(x) = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$. To meet the symmetry requirement, take $\gamma = \beta$.

$$\dot{V}(x) = (\alpha x_1 + \beta x_2)x_2 - (\beta x_1 + \delta x_2)[(x_1 + x_2) + \sin(x_1 + x_2)]$$

Take $\delta = \beta$.

$$\dot{V}(x) = -\beta x_1^2 + (\alpha - 2\beta)x_1 x_2 - \beta(x_1 + x_2)\sin(x_1 + x_2)$$

Taking $\alpha = 2\beta$ and $\beta > 0$ yields

$$\dot{V}(x) = -\beta x_1^2 - \beta(x_1 + x_2)\sin(x_1 + x_2)$$

which is negative definite in the set $\{|x_1 + x_2| < \pi\}$. Now

$$g(x) = \beta \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x \stackrel{\text{def}}{=} Px \;\Rightarrow\; V(x) = \int_0^x g^T(y)\; dy = \frac{1}{2} x^T P x$$

where P is positive definite.

4. For $|2x_1 + x_2| \le 1$, we have

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$$

This matrix is Hurwitz. Hence, the origin is asymptotically stable. It is not globally asymptotically stable because there is another equilibrium point at $(1, 0)$.

5. Take $V(x) = -\,\frac{1}{6}x_1^6 + \frac{1}{4}x_2^4$.

$$\dot{V}(x) = -x_1^5 \dot{x}_1 + x_2^3 \dot{x}_2 = x_1^6 + x_2^6 - x_1^5 x_2^6 + x_2^3 x_1^6$$

Near the origin

$$\left|-x_1^5 x_2^6 + x_2^3 x_1^6\right| \le \tfrac{1}{2}\left(x_1^6 + x_2^6\right)$$

Hence

$$\dot{V}(x) \ge \left(1 - \tfrac{1}{2}\right)\left(x_1^6 + x_2^6\right)$$

which shows that $\dot{V}(x)$ is positive definite. Application of Chetaev's theorem shows that the origin is unstable.

6. Since $zg(z) > 0$ for $z \ne 0$, we have $g(z) > 0$ for $z > 0$ and $g(z) < 0$ for $z < 0$. Since $g(z)$ is continuous, $g(0) = 0$. At the origin $(z = 0,\; y = 0)$, $\sum a_i y_i = 0$, $h(\cdot,\cdot) y_i = 0$, and $g(0) = 0$. Hence, the origin is an equilibrium point. Consider the Lyapunov function candidate

$$V(z, y) = \alpha \int_0^z g(s)\; ds + \sum_{i=1}^m \beta_i y_i^2, \quad \alpha > 0,\; \beta_i > 0$$

$V(z, y)$ is positive definite and radially unbounded.

$$\dot{V} = \alpha g(z)\dot{z} + 2\sum_{i=1}^m \beta_i y_i \dot{y}_i = \sum_{i=1}^m (-\alpha a_i + 2\beta_i b_i) y_i g(z) - 2\sum_{i=1}^m \beta_i h(z, y) y_i^2$$

Choose $\beta_i = \alpha a_i / 2b_i$, to obtain

$$\dot{V} = -2\sum_{i=1}^m \beta_i h(z, y) y_i^2 \le 0, \quad \forall\; (z, y)$$

Thus $\dot{V}$ is negative semidefinite.

$$\dot{V} = 0 \Rightarrow y_i(t) \equiv 0,\; \forall\; i \Rightarrow \dot{y}_i(t) \equiv 0,\; \forall\; i \Rightarrow g(z(t)) \equiv 0 \Rightarrow z(t) \equiv 0$$

By LaSalle's theorem (Corollary 4.2), the origin is globally asymptotically stable.

7.

$$\begin{aligned} 0 &= x_2 \\ 0 &= -x_1 - x_2 \;\text{sat}(x_2^2 - x_3^2) \;\Rightarrow\; x_1 = 0 \\ 0 &= x_3 \;\text{sat}(x_2^2 - x_3^2) \;\Rightarrow\; x_3 \;\text{sat}(-x_3^2) = 0 \;\Rightarrow\; x_3 = 0 \end{aligned}$$

Hence, the origin is the unique equilibrium point. consider

$$\begin{aligned} V(x) &= x^T x \;=\; x_1^2 + x_2^2 + x_3^2 \\ \dot{V}(x) &= 2[x_1x_2 - x_1x_2 - x_2^2\text{sat}(x_2^2 - x_3^2) + x_3^2\text{sat}(x_2^2 - x_3^2)] = -2(x_2^2 - x_3^2)\text{sat}(x_2^2 - x_3^2) \;\le\; 0 \end{aligned}$$

$\dot{V}$ is negative semidefinite.

$$\dot{V} = 0 \Rightarrow x_2^2(t) \equiv x_3^2(t) \Rightarrow \dot{x}_3(t) \equiv 0$$

Hence, both $x_2(t)$ and $x_3(t)$ must be constants. This implies that $\dot{x}_2(t) \equiv 0$. From the second state equation we conclude that $x_1(t) \equiv 0$. Then, the first state equation implies that $x_2(t) \equiv 0$. Consequently, $x_3(t) \equiv 0$. By LaSalle's theorem (Corollary 4.2), the origin is globally asymptotically stable.

8. $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ is positive definite and radially unbounded.

$$\dot{V}(x) = x_1^3[-kh(x)x_1 + x_2] + x_2[-h(x)x_2 - x_1^3] = -kx_1^4 h(x) - x_2^2 h(x)$$

(1) $k > 0$, $h(x) > 0 \;\forall\; x \in D$. In this case $\dot{V}(x)$ is negative definite. Hence, the origin is asymptotically stable.
(2) $k > 0$, $h(x) > 0 \;\forall\; x \in R^2$. In this case $\dot{V}(x) < 0, \;\forall\; x \neq 0$. Hence, the origin is globally asymptotically stable.
(3) $k > 0$, $h(x) < 0 \;\forall\; x \in D$. In this case $\dot{V}(x)$ is positive definite. Hence, by Chetaev's theorem, the origin is unstable.
(4) $k > 0$, $h(x) = 0 \;\forall\; x \in D$. In this case $\dot{V}(x) = 0$. Hence, the origin is stable. It is not asymptotically stable because trajectories starting on the level surface $V(x) = c$ remain on the surface for all future time.
(5) $k = 0$, $h(x) > 0 \;\forall\; x \in D$. In this case $\dot{V}(x) = -x_2^2 h(x) \le 0, \;\forall\; x \in D$. Moreover

$$\dot{V}(x) = 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Hence, by LaSalle's theorem (Corollary 4.1), the origin is asymptotically stable.
(6) $k = 0$, $h(x) > 0 \;\forall\; x \in R^2$. The same as part (5), except that all the conditions hold globally. Hence, the origin is globally asymptotically stable.

9. **(a)**

$$0 = -x_1 + g(x_3), \qquad 0 = -g(x_3), \qquad 0 = -ax_1 + bx_2 - cx_3$$

From the properties of $g(\cdot)$ we know that $g(x_3) = 0$ has an isolated solution $x_3 = 0$. Substituting $x_3 = 0$ in the foregoing equations we obtain $x_1 = x_2 = 0$. Hence, the origin is an isolated equilibrium point.
(b)

$$\begin{aligned} V(x) &= \frac{a}{2}x_1^2 + \frac{b}{2}x_2^2 + \int_0^{x_3} g(y)\;dy \\ \dot{V}(x) &= ax_1[-x_1 + g(x_3)] - bx_2 g(x_3) + g(x_3)[-ax_1 + bx_2 - cg(x_3)] = -ax_1^2 - cg^2(x_3) \;\le\; 0 \end{aligned}$$

$$\dot{V}(x) = 0 \Rightarrow x_1(t) \equiv 0 \text{ and } x_3(t) \equiv 0 \Rightarrow \dot{x}_3(t) \equiv 0$$

From the third state equation we see that $x_2(t) \equiv 0$. Hence, by LaSalle's theorem (Corollary 4.1), the origin is asymptotically stable.
(c) To conclude that the origin is globally asymptotically stable, we need to know that $V(x)$ is radially unbounded. But this is not guaranteed since

$$yg(y) > 0, \;\forall\; |y| \neq 0 \not\Rightarrow \int_0^x g(y)\;dy \to \infty \text{ as } |x| \to \infty.$$

Consider, for example, $g(y) = (1 - e^{-|y|})e^{-|y|}\text{sgn}(y)$. For $x > 0$, we have

$$\int_0^x (1 - e^{-y})e^{-y}\, dy = 1 - e^{-x} - \tfrac{1}{2}(1 - e^{-2x}) \to \tfrac{1}{2} \text{ as } x \to \infty$$

Thus we cannot conclude that the origin is globally asymptotically stable.

10.

$$V(x) = 2a(1 - \cos x_1) + kx_1^2 + x_2^2 + px_3^2 \geq kx_1^2 + x_2^2 + px_3^2$$

Hence, V is positive definite and radially unbounded.

$$\dot{V}(x) = 2(-dx_2^2 - cx_2x_3 - px_3^2 + px_2x_3)$$

Taking $p = c$, we obtain

$$\dot{V} = -2dx_2^2 - 2px_3^2 \leq 0, \ \forall\ x \in R^3$$

$$\dot{V} \equiv 0 \ \Rightarrow \ x_2(t) \equiv 0 \ \&\ x_3(t) \equiv 0 \ \Rightarrow \ a \sin x_1(t) + kx_1(t) \equiv 0$$

Since $k > a$,

$$a \sin x_1(t) + kx_1(t) \equiv 0 \Rightarrow \ x_1(t) \equiv 0$$

Using LaSalle's theorem (Corollary 4.2), we conclude that the origin is globally asymptotically stable.

11.

$$\begin{aligned} 0 &= \frac{1}{1 + x_3} - x_1 \\ 0 &= x_1 - 2x_2 \ \Rightarrow \ x_1 = 2x_2 \\ 0 &= x_2 - 3x_3 \ \Rightarrow \ x_3 = \tfrac{1}{3}x_2 \end{aligned}$$

Substitution of x_1 and x_3 in the first equation yields

$$2x_2^2 + 6x_2 - 3 = 0 \Rightarrow x_2 = \frac{-3 \pm \sqrt{15}}{2}$$

Thus there is only one equilibrium point in the region $x_i \geq 0$; namely,

$$x_1 = \sqrt{15} - 3, \quad x_2 = \frac{\sqrt{15} - 3}{2}, \quad x_3 = \frac{\sqrt{15} - 3}{6}$$

$$\frac{\partial f}{\partial x} = \left.\begin{bmatrix} -1 & 0 & \frac{-1}{(1+x_3)^2} \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix}\right|_{x_3 = \frac{\sqrt{15}-3}{6}} = \begin{bmatrix} -1 & 0 & -0.7621 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

The eigenvalues are $-1.3671 \pm j\ 0.449$ and -3.2657. Hence, the equilibrium point is asymptotically stable.

12. **(1)**

$$\begin{aligned} 0 &= -x_1 + x_2 \ \Rightarrow \ x_1 = x_2 \\ 0 &= (x_1 + x_2)\sin x_1 - 3x_2 \end{aligned}$$

Thus

$$x_1(2\sin x_1 - 3) = 0 \Rightarrow x_1 = 0 \Rightarrow x_2 = 0$$

Hence, the origin is the unique equilibrium point.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 1 \\ \sin x_1 + (x_1 + x_2)\cos x_1 & \sin x_1 - 3 \end{bmatrix}; \quad A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$$

A is Hurwitz; hence, the origin is asymptotically stable. To show global asymptotic stability, let $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\begin{aligned} \dot{V}(x) &= -x_1^2 + x_1 x_2(1 + \sin x_1) - (3 - \sin x_1)x_2^2 \leq -x_1^2 + 2|x_1|\,|x_2| - 2x_2^2 \\ &= -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} < 0, \quad \forall\, x \neq 0 \end{aligned}$$

Hence, the origin is globally asymptotically stable.

(2)

$$\begin{aligned} 0 &= -x_1^3 + x_2 \\ 0 &= -ax_1 - bx_2 \;\Rightarrow\; x_2 = -\frac{b}{a}x_1 \;\Rightarrow\; -x_1(x_1^2 + a/b) = 0 \;\Rightarrow\; x_1 = 0 \end{aligned}$$

Hence, the origin is the unique equilibrium point.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -3x_1^2 & 1 \\ -a & -b \end{bmatrix}; \quad A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

A is Hurwitz; hence, the origin is asymptotically stable. To show global asymptotic stability, let $V(x) = \frac{1}{2}(x_1^2 + \alpha x_2^2)$, $\alpha > 0$.

$$\dot{V}(x) = -x_1^4 + x_1 x_2(1 - a\alpha) - b\alpha x_2^2$$

Taking $\alpha = 1/a$, we obtain

$$\dot{V}(x) = -x_1^4 - \frac{b}{a}x_2^2 < 0, \quad \forall\, x \neq 0$$

Hence, the origin is globally asymptotically stable.

13. Take $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\dot{V} = x_1[-x_1^3 + \alpha(t)x_2] + x_2[-\alpha(t)x_1 - x_2^3] = -x_1^4 - x_2^4$$

Hence, the origin is globally uniformly asymptotically stable. Linearization at the origin yields

$$A(t) = \begin{bmatrix} 0 & \alpha(t) \\ -\alpha(t) & 0 \end{bmatrix}$$

The origin of the linear system is not exponentially stable since, with $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$, we have $\dot{V} = 0$ which implies that $V(x(t))$ is constant along the solution. Thus, $x(t)$ does not converge to the origin as t tends to infinity. From Theorem 4.15, we conclude that the origin of the nonlinear system is not exponentially stable.

14. View this system as a perturbation of the autonomous system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2$$

The matrix A of the autonomous system is Hurwitz. Find a Lyapunov function $V(x) = x^T P x$ for the autonomous system by solving the Lyapunov equation $PA + A^T P = -I$, to obtain $P = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

Now use $V(x) = x^T P x = \frac{3}{2}x_1^2 + x_1 x_2 + x_2^2$ as a Lyapunov function candidate for the perturbed system with $b \neq 0$.

$$\begin{aligned} \dot{V} &= -x_1^2 - x_2^2 - b\cos t\ x_1 x_2 - 2b\cos t\ x_2^2 \ \leq\ -x_1^2 - x_2^2 + |b|\ |x_1|\ |x_2| + 2|b|\ |x_2|^2 \\ &= -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -|b|/2 \\ -|b|/2 & 1-2|b| \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \end{aligned}$$

The right hand side is negative definite if $(1 - 2|b| - b^2/4) > 0$, which is the case if $|b| < 0.472$. Taking $b^* = 0.472$, we conclude that the origin is exponentially stable for all $|b| < b^*$.

15. Try $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$$\dot{V} = x_1 x_2 - g(t)x_1^2(x_1^2 + x_2^2) - x_1 x_2 - g(t)x_2^2(x_1^2 + x_2^2) = -g(t)(x_1^2 + x_2^2)^2 \leq -4kV^2(x)$$

Hence, $\dot{V}(t,x)$ is negative definite and the origin is uniformly asymptotically stable. From the inequality $\dot{V} \leq -4kV^2$, we cannot conclude exponential stability. Let us try linearization.

$$A(t) = \left.\frac{\partial f}{\partial x}(t,x)\right|_{x=0} = \left.\begin{bmatrix} -3g(t)x_1^2 - g(t)x_2^2 & 1 - 2g(t)x_1 x_2 \\ -1 - 2g(t)x_1 x_2 & -3g(t)x_2^2 - g(t)x_1^2 \end{bmatrix}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

A is a constant matrix that is not Hurwitz. Hence, by Theorem 4.15, we conclude that the origin is not exponentially stable.

16. Let $A_1 = \frac{\partial f}{\partial x}(0)$ be the linearization of (1). To find the linearization of (2), set $g(x) = h(x)f(x)$. Then

$$\frac{\partial g_i}{\partial x_j} = h(x)\frac{\partial f_i}{\partial x_j} + \frac{\partial h}{\partial x_j} f_i(x)$$

Hence

$$\frac{\partial g_i}{\partial x_j}(0) = h(0)\frac{\partial f_i}{\partial x_j}(0) + \frac{\partial h}{\partial x_j}(0) f_i(0) = h(0)\frac{\partial f_i}{\partial x_j}(0)$$

Hence

$$A_2 = \frac{\partial g}{\partial x}(0) = h(0)A_1$$

Since $h(0) > 0$, A_1 is Hurwitz if and only if A_2 is Hurwitz. By Theorem 4.15

$$\text{(1) is exp. stable} \ \Leftrightarrow\ A_1 \text{ is Hurwitz} \ \Leftrightarrow\ A_2 \text{ is Hurwitz} \ \Leftrightarrow\ \text{(2) is exp. stable}$$

17. It is not input-to-state stable because with $u = 0$ the origin is not globally asymptotically stable (notice that the unforced system has equilibrium points on the unit circle).