Additional Exercises for Chapter 4

1. For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Then, investigate whether the origin is globally asymptotically stable.

$$\textbf{(1)}\ \ \dot{x}_1 = -x_1 + x_2^2, \qquad \dot{x}_2 = -x_2$$

$$\textbf{(2)}\ \ \dot{x}_1 = (x_1 - x_2)(x_1^2 + x_2^2 - 1), \qquad \dot{x}_2 = (x_1 + x_2)(x_1^2 + x_2^2 - 1)$$

$$\textbf{(3)}\ \ \dot{x}_1 = -x_1 + x_1^2 x_2, \qquad \dot{x}_2 = -x_2 + x_1$$

2. Using $V(x) = x_1^2 + x_2^2$, study stability of the origin of the system

$$\dot{x}_1 = x_1(k^2 - x_1^2 - x_2^2) + x_2(x_1^2 + x_2^2 + k^2), \qquad \dot{x}_2 = -x_1(k^2 + x_1^2 + x_2^2) + x_2(k^2 - x_1^2 - x_2^2)$$

when (a) $k = 0$ and (b) $k \neq 0$.

3. Using the variable gradient method, find a Lyapunov function $V(x)$ that shows asymptotic stability of the origin of the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(x_1 + x_2) - \sin(x_1 + x_2)$$

4. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 - \text{sat}(2x_1 + x_2)$$

Show that the origin is asymptotically stable, but not globally asymptotically stable.

5. Show that the origin of the following system is unstable.

$$\dot{x}_1 = \quad -x_1 + x_2^6, \qquad \dot{x}_2 = x_2^3 + x_1^6$$

6. Consider the system

$$\dot{z} = -\sum_{i=1}^{m} a_i y_i, \qquad \dot{y}_i = -h(z,y)y_i + b_i g(z), \quad i = 1, 2, \ldots, m$$

where z is a scalar, $y^T = (y_1, \ldots, y_m)$. The functions $h(\cdot,\cdot)$ and $g(\cdot)$ are continuously differentiable for all (z, y) and satisfy $zg(z) > 0,\ \forall\ z \neq 0$, $h(z, y) > 0,\ \forall\ (z, y) \neq 0$, and $\int_0^z g(\sigma)\ d\sigma \to \infty$ as $|z| \to \infty$. The constants a_i and b_i satisfy $b_i \neq 0$ and $a_i/b_i > 0,\ \forall\ i = 1, 2, \ldots, m$. Show that the origin is an equilibrium point, and investigate its stability using a Lyapunov function candidate of the form

$$V(z, y) = \alpha \int_0^z g(\sigma)\ d\sigma + \sum_{i=1}^{m} \beta_i y_i^2$$

7. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2\ \text{sat}(x_2^2 - x_3^2), \qquad \dot{x}_3 = x_3\ \text{sat}(x_2^2 - x_3^2)$$

where $\text{sat}(\cdot)$ is the saturation function. Show that the origin is the unique equilibrium point, and use $V(x) = x^T x$ to show that it is globally asymptotically stable.

8. The origin $x = 0$ is an equilibrium point of the system

$$\dot{x}_1 = -kh(x)x_1 + x_2, \qquad \dot{x}_2 = -h(x)x_2 - x_1^3$$

Let $D = \{x \in R^2 \mid \|x\|_2 < 1\}$. Using $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, investigate stability of the origin in each of the following cases.
(1) $k > 0,\ \ h(x) > 0, \forall\ x \in D$; **(2)** $k > 0,\ \ h(x) > 0, \forall\ x \in R^2$;
(3) $k > 0,\ \ h(x) < 0, \forall\ x \in D$; **(4)** $k > 0,\ \ h(x) = 0, \forall\ x \in D$;
(5) $k = 0,\ \ h(x) > 0, \forall\ x \in D$; **(6)** $k = 0,\ \ h(x) > 0, \forall\ x \in R^2$.

9. Consider the system

$$\dot{x}_1 = -x_1 + g(x_3), \qquad \dot{x}_2 = -g(x_3), \qquad \dot{x}_3 = -ax_1 + bx_2 - cg(x_3)$$

where a, b, and c are positive constants and $g(\cdot)$ is a locally Lipschitz function that satisfies

$$g(0) = 0 \ \text{ and } \ yg(y) > 0, \ \ \forall\ 0 < |y| < k, \ \ k > 0$$

(a) Show that the origin is an isolated equilibrium point.

(b) With $V(x) = \frac{1}{2}ax_1^2 + \frac{1}{2}bx_2^2 + \int_0^{x_3} g(y)\ dy$ as a Lyapunov function candidate, show that the origin is asymptotically stable.

(c) Suppose $yg(y) > 0\ \forall\ y \neq 0$. Is the origin globally asymptotically stable?

10. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a\sin x_1 - kx_1 - dx_2 - cx_3, \qquad \dot{x}_3 = -x_3 + x_2$$

where all coefficients are positive and $k > a$. Using $V(x) = 2a\int_0^{x_1} \sin y\ dy + kx_1^2 + x_2^2 + px_3^2$ with some $p > 0$, show that the origin is globally asymptotically stable.

11. Show that the system

$$\dot{x}_1 = \frac{1}{1 + x_3} - x_1, \qquad \dot{x}_2 = x_1 - 2x_2, \qquad \dot{x}_3 = x_2 - 3x_3$$

has a unique equilibrium point in the region $x_i \geq 0$, $i = 1, 2, 3$, and investigate stability of this point using linearization.

12. For each of the following systems, use linearization to show that the origin is asymptotically stable. Then, show that the origin is globally asymptotically stable.

$$(\mathbf{1}) \quad \begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= (x_1 + x_2)\sin x_1 - 3x_2 \end{aligned} \qquad\qquad (\mathbf{2}) \quad \begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= -ax_1 - bx_2, \ a, b > 0 \end{aligned}$$

13. Consider the system

$$\dot{x}_1 = -x_1^3 + \alpha(t)x_2, \qquad \dot{x}_2 = -\alpha(t)x_1 - x_2^3$$

where $\alpha(t)$ is a continuous, bounded function. Show that the origin is globally uniformly asymptotically stable. Is it exponentially stable?

14. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - (1 + b\cos t)x_2$$

Find $b^* > 0$ such that the origin is exponentially stable for all $|b| < b^*$.

15. Consider the system

$$\dot{x}_1 = x_2 - g(t)x_1(x_1^2 + x_2^2), \qquad \dot{x}_2 = -x_1 - g(t)x_2(x_1^2 + x_2^2)$$

where $g(t)$ is continuously differentiable, bounded, and $g(t) \geq k > 0$ for all $t \geq 0$. Is the origin uniformly asymptotically stable? Is it exponentially stable?

16. Consider two systems represented by

$$\dot{x} = f(x) \qquad (1)$$

$$\dot{x} = h(x)f(x) \qquad (2)$$

where $f : R^n \to R^n$ and $h : R^n \to R$ are continuously differentiable, $f(0) = 0$, and $h(0) > 0$. Show that the origin of (1) is exponentially stable if and only if the origin of (2) is exponentially stable.

17. Investigate input-to-state stability of the system

$$\dot{x}_1 = (x_1 - x_2 + u)(x_1^2 + x_2^2 - 1), \qquad \dot{x}_2 = (x_1 + x_2 + u)(x_1^2 + x_2^2 - 1)$$