Solution of Additional Exercises for Chapter 2

1. **(1)**

$$0 = x_2, \qquad 0 = -x_1 + \frac{1}{6}x_1^3 - x_2 \;\Rightarrow\; x_1 = 0, \sqrt{6}, -\sqrt{6}$$

There are three equilibrium points at $(0,0)$, $(\sqrt{6},0)$, $(-\sqrt{6},0)$.

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{6}x_1^3 - x_2 \end{bmatrix}, \quad \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + \frac{1}{2}x_1^2 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}; \quad \text{Eigenvalues} : -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$(0,0)$ is a stable focus.

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=\sqrt{6}} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}; \quad \text{Eigenvalues} : 1, -2$$

$(\sqrt{6},0)$ is a saddle. Similarly, $(-\sqrt{6},0)$ is a saddle.

(2)

$$0 = -x_1 + x_2, \qquad 0 = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3$$

Hence $x_2 = x_1$ and

$$0 = x_1(1.9 + x_1 + 0.1x_1^2) \;\Rightarrow\; x_1 = 0,\; -2.5505,\; \text{or} \; -7.4495$$

There are three equilibrium points at $(0,0)$, $(-2.5505,-2.5505)$, and $(-7.4495,-7.4495)$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 1 \\ 0.1 - 2x_1 - 0.3x_1^2 & -2 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0} = \begin{bmatrix} -1 & 1 \\ 0.1 & -2 \end{bmatrix}; \quad \text{Eigenvalues} : -0.9084, -2.0916$$

$(0,0)$ is a stable node.

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=-2.5505} = \begin{bmatrix} -1 & 1 \\ 3.2495 & -2 \end{bmatrix}; \quad \text{Eigenvalues} : 0.3707, -3.3707$$

$(-2.5505,-2.5505)$ is a saddle point.

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=-7.4495} = \begin{bmatrix} -1 & 1 \\ -1.8695 & -2 \end{bmatrix}; \quad \text{Eigenvalues} : -1.5 \pm j1.183$$

$(-7.4495,-7.4495)$ is a stable focus.

(3)

$$0 = x_2, \qquad 0 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

There is a unique equilibrium point at $(0,0)$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 - 6x_1x_2 & 1 - 3x_1^2 - 6x_2^2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Eigenvalues are $1/2 \pm j\sqrt{3}/2$; hence, $(0,0)$ is unstable focus.

(4)

$$0 = -x_1 + x_2(1 + x_1), \qquad 0 = -x_1(1 + x_1) \;\Rightarrow\; x_1 = 0 \text{ or } x_1 = -1$$

At $x_1 = 0$, $x_2 = 0$. At $x_1 = -1$, $\dot{x}_1 = 1 \neq 0$. Hence, there is a unique equilibrium point at $(0,0)$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} -1 + x_2 & 1 + x_1 \\ -1 + 2x_1 & 0 \end{bmatrix}_{(0,0)} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues are $-1/2 \pm j\sqrt{3}/2$; hence, $(0,0)$ is stable focus.

2.

$$0 = ax_1 - x_1x_2 = x_1(a - x_2), \qquad 0 = bx_1^2 - cx_2$$

From the first equation, $x_1 = 0$ or $x_2 = a$. From the second equation, $x_1 = 0 \Rightarrow x_2 = 0$, and $x_2 = a \Rightarrow x_1 = \pm\sqrt{ac/b}$. Thus there are three equilibrium points at $(0,0)$, $(\sqrt{ac/b}, a)$ and $(-\sqrt{ac/b}, a)$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} a - x_2 & -x_1 \\ 2bx_1 & -c \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

The eigenvalues are $a > 0$ and $-c < 0$; $(0,0)$ is a saddle.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(\sqrt{ac/b},a)} = \begin{bmatrix} 0 & -\sqrt{ac/b} \\ 2\sqrt{abc} & -c \end{bmatrix}$$

The eigenvalues are

$$\frac{-c \pm \sqrt{c^2 - 8ac}}{2}$$

The equilibrium point $(\sqrt{ac/b}, a)$ is a stable node if $8a < c$ and a stable focus if $8a > c$. The same is true for the equilibrium point $(-\sqrt{ac/b}, a)$.

3. **(1)** The system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 - 2\tan^{-1}(x_1 + x_2)$$

has three equilibrium points at $(0,0)$, $(a,0)$, and $(-a,0)$, where a is the root of $a - \tan\left(\frac{a}{2}\right) = 0$; $a \approx 2.33$. The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 1 - \frac{2}{1+(x_1+x_2)^2} & \frac{-2}{1+(x_1+x_2)^2} \end{bmatrix}$$

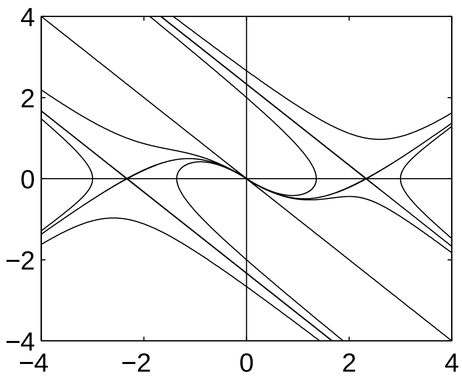

Figure 1: Exercise 3(1).

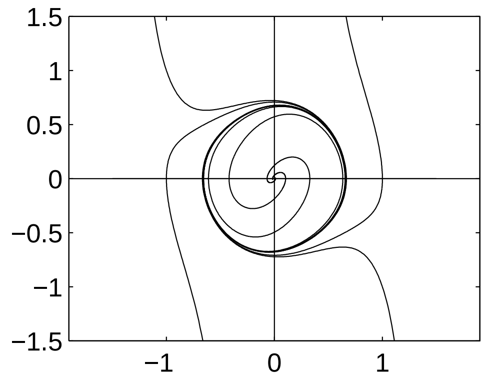

Figure 2: Exercise 3(2).

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0;x_2=0} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \Rightarrow \text{ Eigenvalues } = -1\& -1 \Rightarrow \ (0,0) \text{ is a stable node}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=\pm 2.33;x_2=0} = \begin{bmatrix} 0 & 1 \\ 0.6889 & -0.3111 \end{bmatrix} \Rightarrow \text{ Eigenvalues } = 0.6889\& -1 \Rightarrow \ (\pm 2.33,0) \text{ is a saddle}$$

The phase portrait is shown in Figure 1. The portrait shows separatrices formed by the stable trajectories of the saddle points. The separatrices define a layer. Every trajectory starting inside this layer approaches the stable node as $t \to \infty$. Trajectories starting outside this layer approach infinity as $t \to \infty$. It is noted also that the line $x_2 = -x_1$ is a trajectory approaching the origin. This behavior can be also seen by defining a new variable $\sigma = x_1 + x_2$ and noting that σ satisfies the scalar equation $\dot{\sigma} = \sigma - 2\tan^{-1}\sigma$.

(2) The system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$$

has one equilibrium point at the origin. The Jacobian at the origin is given by

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0;x_2=0} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \text{ Eigenvalues } = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

Hence the origin is unstable focus. The phase portrait is shown in Figure 2. The system has a stable limit cycle. All trajectories, except the trivial solution $x = 0$, spiral toward the limit cycle.

(3) The system

$$\dot{x}_1 = x_1(2 - x_2), \qquad \dot{x}_2 = 2x_1^2 - x_2$$

has three equilibrium points at $(0,0)$, $(1,2)$, and $(-1,2)$. The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2 - x_2 & -x_1 \\ 4x_1 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0;x_2=0} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{ Eigenvalues } = 2\& -1 \Rightarrow \ (0,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=1;x_2=2} = \begin{bmatrix} 0 & -1 \\ 4 & -1 \end{bmatrix} \Rightarrow \text{ Eigenvalues } = -0.5 \pm j1.94 \Rightarrow \ (1,2) \text{ is a stable focus}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=-1;x_2=2} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \Rightarrow \text{ Eigenvalues } = -0.5 \pm j1.94 \Rightarrow \ (-1,2) \text{ is a stable focus}$$

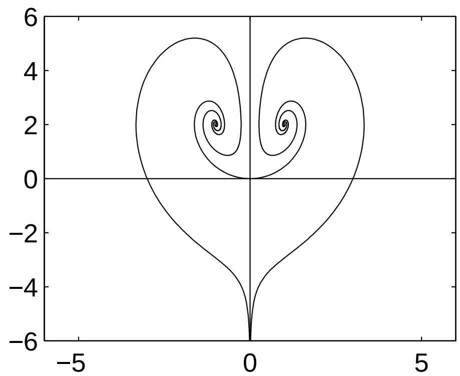

Figure 3: Exercise 3(3).

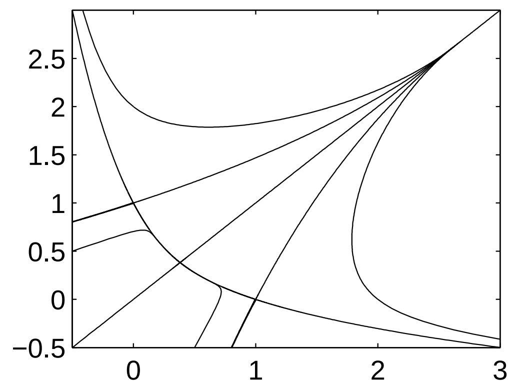

Figure 4: Exercise 4.

The phase portrait is shown in Figure 3. The stable trajectories which approach the saddle point move along the x_2-axis. Therefore, the x_2-axis forms a separatrix which divides the plane into two haves. Trajectories in each half spiral toward the stable focus in the respective halves.

4. (a) The equilibrium points are given by the real roots of the equation

$$0 = y^4 - 2y^2 + y$$

where $x_1 = y^2$ and $x_2 = 1 - y$. It can be seen that the equation has four roots at $y = 0, 1, (-1 \pm \sqrt{5})/2$. Hence, there are four equilibrium points at $(0, 1)$, $(1, 0)$, $((3 - \sqrt{5})/2, (3 - \sqrt{5})/2)$, and $((3 + \sqrt{5})/2, (3 + \sqrt{5})/2)$. The following table shows the Jacobian matrix and the type of each point.

Point	Jacobian matrix	Eigenvalues	Type
$(0, 1)$	$\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$	$-2.4142,\ 0.4142$	saddle
$(1, 0)$	$\begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$	$0.4142,\ -2.4142$	saddle
$((3 - \sqrt{5})/2, (3 - \sqrt{5})/2)$	$\begin{bmatrix} -1.2361 & -1 \\ -1 & -1.2361 \end{bmatrix}$	$-0.2361,\ -2.2361$	stable node
$((3 + \sqrt{5})/2, (3 + \sqrt{5})/2)$	$\begin{bmatrix} 3.2361 & -1 \\ -1 & 3.2361 \end{bmatrix}$	$4.2361,\ 2.2361$	unstable node

(b) The phase portrait is shown in Figure 4. The stable trajectories of the saddle points divide the plane into two regions. The region that contains the stable focus has the feature that all trajectories inside it converge to the stable focus. All trajectories in the other region diverge to infinity.