Basic Concepts

We want to stabilize the system

\[ \dot{x} = f(x, u) \]

at the equilibrium point \( x = x_{ss} \)

**Steady-State Problem:** Find steady-state control \( u_{ss} \) s.t.

\[ 0 = f(x_{ss}, u_{ss}) \]

\[ x_{\delta} = x - x_{ss}, \quad u_{\delta} = u - u_{ss} \]

\[ \dot{x}_{\delta} = f(x_{ss} + x_{\delta}, u_{ss} + u_{\delta}) \overset{\text{def}}{=} f_{\delta}(x_{\delta}, u_{\delta}) \]

\[ f_{\delta}(0, 0) = 0 \]

\[ u_{\delta} = \phi(x_{\delta}) \Rightarrow u = u_{ss} + \phi(x - x_{ss}) \]
State Feedback Stabilization: Given

\[
\dot{x} = f(x, u) \quad [f(0, 0) = 0]
\]

find

\[
u = \phi(x) \quad [\phi(0) = 0]
\]

s.t. the origin is an asymptotically stable equilibrium point of

\[
\dot{x} = f(x, \phi(x))
\]

\(f\) and \(\phi\) are locally Lipschitz functions
Notions of Stabilization

\[ \dot{x} = f(x, u), \quad u = \phi(x) \]

Local Stabilization: The origin of \( \dot{x} = f(x, \phi(x)) \) is asymptotically stable (e.g., linearization)

Regional Stabilization: The origin of \( \dot{x} = f(x, \phi(x)) \) is asymptotically stable and a given region \( G \) is a subset of the region of attraction (for all \( x(0) \in G \), \( \lim_{t \to \infty} x(t) = 0 \)) (e.g., \( G \subset \Omega_c = \{ V(x) \leq c \} \) where \( \Omega_c \) is an estimate of the region of attraction)

Global Stabilization: The origin of \( \dot{x} = f(x, \phi(x)) \) is globally asymptotically stable
Semiglobal Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable and $\phi(x)$ can be designed such that any given compact set (no matter how large) can be included in the region of attraction (Typically $u = \phi_p(x)$ is dependent on a parameter $p$ such that for any compact set $G$, $p$ can be chosen to ensure that $G$ is a subset of the region of attraction).

What is the difference between global stabilization and semiglobal stabilization?
Example 9.1

\[
\dot{x} = x^2 + u
\]

Linearization:

\[
\dot{x} = u, \quad u = -kx, \quad k > 0
\]

Closed-loop system:

\[
\dot{x} = -kx + x^2
\]

Linearization of the closed-loop system yields \( \dot{x} = -kx \). Thus, \( u = -kx \) achieves local stabilization.

The region of attraction is \( \{x < k\} \). Thus, for any set \( \{-a \leq x \leq b\} \) with \( b < k \), the control \( u = -kx \) achieves regional stabilization.
The control $u = -kx$ does not achieve global stabilization.

But it achieves semiglobal stabilization because any compact set $\{|x| \leq r\}$ can be included in the region of attraction by choosing $k > r$.

The control

$$u = -x^2 - kx$$

achieves global stabilization because it yields the linear closed-loop system $\dot{x} = -kx$ whose origin is globally exponentially stable.
Linearization

\[ \dot{x} = f(x, u) \]

\( f(0, 0) = 0 \) and \( f \) is continuously differentiable in a domain \( D_x \times D_u \) that contains the origin \( (x = 0, u = 0) \) \( (D_x \subset \mathbb{R}^n, D_u \subset \mathbb{R}^m) \)

\[ \dot{x} = Ax + Bu \]

\[ A = \left. \frac{\partial f}{\partial x}(x, u) \right|_{x=0,u=0} ; \quad B = \left. \frac{\partial f}{\partial u}(x, u) \right|_{x=0,u=0} \]

Assume \( (A, B) \) is stabilizable. Design a matrix \( K \) such that \( (A - BK) \) is Hurwitz

\[ u = -Kx \]
Closed-loop system:

\[ \dot{x} = f(x, -Kx) \]

Linearization:

\[ \dot{x} = \left[ \frac{\partial f}{\partial x}(x, -Kx) + \frac{\partial f}{\partial u}(x, -Kx)(-K) \right] x \]
\[ = (A - BK)x \]

Since \((A - BK)\) is Hurwitz, the origin is an exponentially stable equilibrium point of the closed-loop system.
Feedback Linearization

Consider the nonlinear system

\[ \dot{x} = f(x) + G(x)u \]

\[ f(0) = 0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \]

Suppose there is a change of variables \( z = T(x) \), defined for all \( x \in D \subset \mathbb{R}^n \), that transforms the system into the controller form

\[ \dot{z} = Az + B[\psi(x) + \gamma(x)u] \]

where \((A, B)\) is controllable and \( \gamma(x) \) is nonsingular for all \( x \in D \)

\[ u = \gamma^{-1}(x)[-\psi(x) + v] \quad \Rightarrow \quad \dot{z} = Az + Bv \]
\[ u = -Kz \]

Design \( K \) such that \((A - BK)\) is Hurwitz

The origin \( z = 0 \) of the closed-loop system

\[ \dot{z} = (A - BK)z \]

is globally exponentially stable

\[ u = \gamma^{-1}(x)[-\psi(x) - KT(x)] \]

Closed-loop system in the \( x \)-coordinates:

\[ \dot{x} = f(x) + G(x)\gamma^{-1}(x)[-\psi(x) - KT(x)] \overset{\text{def}}{=} f_c(x) \]
What can we say about the stability of \( x = 0 \) as an equilibrium point of \( \dot{x} = f_c(x) \)?

\[
z = T(x) \Rightarrow \frac{\partial T}{\partial x}(x)f_c(x) = (A - BK)T(x)
\]

\[
\frac{\partial f_c}{\partial x}(0) = J^{-1}(A - BK)J, \quad J = \frac{\partial T}{\partial x}(0) \text{ (nonsingular)}
\]

The origin of \( \dot{x} = f_c(x) \) is exponentially stable.

Is \( x = 0 \) globally asymptotically stable? In general No.

It is globally asymptotically stable if \( T(x) \) is a global diffeomorphism.
What information do we need to implement the control

\[ u = \gamma^{-1}(x)[-\psi(x) - KT(x)] \]

What is the effect of uncertainty in \( \psi \), \( \gamma \), and \( T \)?

Let \( \hat{\psi}(x) \), \( \hat{\gamma}(x) \), and \( \hat{T}(x) \) be nominal models of \( \psi(x) \), \( \gamma(x) \), and \( T(x) \)

\[ u = \hat{\gamma}^{-1}(x)[-\hat{\psi}(x) - K\hat{T}(x)] \]

Closed-loop system:

\[ \dot{z} = (A - BK)z + B\Delta(\dot{z}) \]
\[
\dot{z} = (A - BK)z + B\Delta(z) \quad (*)
\]

\[
V(z) = z^T Pz, \quad P(A - BK) + (A - BK)^T P = -I
\]

**Lemma 9.1**

Suppose (*) is defined in \( D_z \subset \mathbb{R}^n \)

- If \( \|\Delta(z)\| \leq k\|z\| \quad \forall \ z \in D_z \), \( k < 1/(2\|PB\|) \), then the origin of (*) is exponentially stable. It is globally exponentially stable if \( D_z = \mathbb{R}^n \)

- If \( \|\Delta(z)\| \leq k\|z\| + \delta \quad \forall \ z \in D_z \) and \( B_r \subset D_z \), then there exist positive constants \( c_1 \) and \( c_2 \) such that if \( \delta < c_1 r \) and \( z(0) \in \{ z^T Pz \leq \lambda_{\text{min}}(P)r^2 \} \), \( \|z(t)\| \) will be ultimately bounded by \( \delta c_2 \). If \( D_z = \mathbb{R}^n \), \( \|z(t)\| \) will be globally ultimately bounded by \( \delta c_2 \) for any \( \delta > 0 \).
Example 9.4 (Pendulum Equation)

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin(x_1 + \delta_1) - bx_2 + cu \]

\[ u = \left( \frac{1}{c} \right) \sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2) \]

\[ A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2 + b) \end{bmatrix} \]

\[ u = \left( \frac{1}{\hat{c}} \right) \sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2) \]

\[ \Delta(x) = \left( \frac{c - \hat{c}}{\hat{c}} \right) \sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2) \]
\[ |\Delta(x)| \leq k\|x\| + \delta, \quad \forall x \]

\[ k = \left\| \frac{c - \hat{c}}{\hat{c}} \right\| \left(1 + \sqrt{k_1^2 + k_2^2}\right), \quad \delta = \left\| \frac{c - \hat{c}}{\hat{c}} \right\| \left| \sin \delta_1 \right| \]

\[ P(A - BK) + (A - BK)^T P = -I, \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \]

\[ k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}} \quad \Rightarrow \quad \text{GUB} \]

\[ \sin \delta_1 = 0 \quad \& \quad k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}} \quad \Rightarrow \quad \text{GES} \]
Is feedback linearization a good idea?

Example 9.5

\[ \dot{x} = ax - bx^3 + u, \quad a, b > 0 \]

\[ u = -(k + a)x + bx^3, \quad k > 0, \quad \Rightarrow \quad \dot{x} = -kx \]

\(-bx^3\) is a damping term. Why cancel it?

\[ u = -(k + a)x, \quad k > 0, \quad \Rightarrow \quad \dot{x} = -kx - bx^3 \]

Which design is better?
Example 9.6

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -h(x_1) + u \\
h(0) &= 0 \text{ and } x_1 h(x_1) > 0, \forall x_1 \neq 0
\end{align*}
\]

Feedback Linearization:

\[
u = h(x_1) - (k_1 x_1 + k_2 x_2)
\]

With \( y = x_2 \), the system is passive with

\[
V = \int_0^{x_1} h(z) \, dz + \frac{1}{2} x_2^2
\]

\[
\dot{V} = h(x_1) \dot{x}_1 + x_2 \dot{x}_2 = yu
\]
The control

\[ u = -\sigma(x_2), \quad \sigma(0) = 0, \ x_2\sigma(x_2) > 0 \ \forall \ x_2 \neq 0 \]

creates a feedback connection of two passive systems with storage function \( V \)

\[ \dot{V} = -x_2\sigma(x_2) \]

\[ x_2(t) \equiv 0 \ \Rightarrow \ \dot{x}_2(t) \equiv 0 \ \Rightarrow \ h(x_1(t)) \equiv 0 \ \Rightarrow \ x_1(t) \equiv 0 \]

Asymptotic stability of the origin follows from the invariance principle

Which design is better?
The control \( u = -\sigma(x_2) \) has two advantages:

- It does not use a model of \( h \)
- The flexibility in choosing \( \sigma \) can be used to reduce \( |u| \)

However, \( u = -\sigma(x_2) \) cannot arbitrarily assign the rate of decay of \( x(t) \). Linearization of the closed-loop system at the origin yields the characteristic equation

\[
s^2 + \sigma'(0)s + h'(0) = 0
\]

One of the two roots cannot be moved to the left of \( \text{Re}[s] = -\sqrt{h'(0)} \)
Partial Feedback Linearization

Consider the nonlinear system

\[ \dot{x} = f(x) + G(x)u \quad [f(0) = 0] \]

Suppose there is a change of variables

\[ z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix} \]

defined for all \( x \in D \subset \mathbb{R}^n \), that transforms the system into

\[ \dot{\eta} = f_0(\eta, \xi), \quad \dot{\xi} = A\xi + B[\psi(x) + \gamma(x)u] \]

\((A, B)\) is controllable and \( \gamma(x) \) is nonsingular for all \( x \in D \)
\[ u = \gamma^{-1}(x)[-\psi(x) + v] \]
\[ \dot{\eta} = f_0(\eta, \xi), \quad \dot{\xi} = A\xi + Bv \]
\[ v = -K\xi, \quad \text{where } (A - BK) \text{ is Hurwitz} \]
**Lemma 9.2**

The origin of the cascade connection

\[ \dot{\eta} = f_0(\eta, \xi), \quad \dot{\xi} = (A - BK)\xi \]

is asymptotically (exponentially) stable if the origin of \( \dot{\eta} = f_0(\eta, 0) \) is asymptotically (exponentially) stable

**Proof**

With \( b > 0 \) sufficiently small,

\[
V(\eta, \xi) = bV_1(\eta) + \sqrt{\xi^TP\xi}, \quad (\text{asymptotic})
\]

\[
V(\eta, \xi) = bV_1(\eta) + \xi^TP\xi, \quad (\text{exponential})
\]
If the origin of $\dot{\eta} = f_0(\eta, 0)$ is globally asymptotically stable, will the origin of

$$\dot{\eta} = f_0(\eta, \xi), \quad \dot{\xi} = (A - BK)\xi$$

be globally asymptotically stable?

In general **No**

**Example 9.7**

The origin of $\dot{\eta} = -\eta$ is globally exponentially stable, but

$$\dot{\eta} = -\eta + \eta^2 \xi, \quad \dot{\xi} = -k\xi, \quad k > 0$$

has a finite region of attraction $\{\eta\xi < 1 + k\}$
Example 9.8

\[ \dot{\eta} = -\frac{1}{2}(1 + \xi_2)\eta^3, \quad \dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = v \]

The origin of \( \dot{\eta} = -\frac{1}{2}\eta^3 \) is globally asymptotically stable

\[ v = -k^2\xi_1 - 2k\xi_2 \overset{\text{def}}{=} -K\xi \quad \Rightarrow \quad A - BK = \begin{bmatrix} 0 & 1 \\ -k^2 & -2k \end{bmatrix} \]

The eigenvalues of \((A - BK)\) are \(-k\) and \(-k\)

\[ e^{(A-BK)t} = \begin{bmatrix} (1 + kt)e^{-kt} & te^{-kt} \\ -k^2te^{-kt} & (1 - kt)e^{-kt} \end{bmatrix} \]
Peaking Phenomenon:

\[
\max_t \left\{ k^2 t e^{-kt} \right\} = \frac{k}{e} \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty
\]

\[
\xi_1(0) = 1, \quad \xi_2(0) = 0 \quad \Rightarrow \quad \xi_2(t) = -k^2 t e^{-kt}
\]

\[
\dot{\eta} = -\frac{1}{2} \left( 1 - k^2 t e^{-kt} \right) \eta^3, \quad \eta(0) = \eta_0
\]

\[
\eta^2(t) = \frac{\eta_0^2}{1 + \eta_0^2 [t + (1 + kt)e^{-kt} - 1]}
\]

If \( \eta_0^2 > 1 \), the system will have a finite escape time if \( k \) is chosen large enough.
Lemma 9.3

The origin of

\[ \dot{\eta} = f_0(\eta, \xi), \quad \dot{\xi} = (A - BK)\xi \]

is globally asymptotically stable if the system \( \dot{\eta} = f_0(\eta, \xi) \) is input-to-state stable

Proof

Apply Lemma 4.6

Model uncertainty can be handled as in the case of feedback linearization
Backstepping

\[
\begin{align*}
\dot{\eta} &= f_a(\eta) + g_a(\eta)\xi \\
\dot{\xi} &= f_b(\eta, \xi) + g_b(\eta, \xi)u, \quad g_b \neq 0, \quad \eta \in R^n, \quad \xi, u \in R
\end{align*}
\]

Stabilize the origin using state feedback

View $\xi$ as “virtual” control input to the system

\[
\dot{\eta} = f_a(\eta) + g_a(\eta)\xi
\]

Suppose there is $\xi = \phi(\eta)$ that stabilizes the origin of

\[
\dot{\eta} = f_a(\eta) + g_a(\eta)\phi(\eta)
\]

\[
\frac{\partial V}{\partial \eta} [f_a(\eta) + g_a(\eta)\phi(\eta)] \leq -W(\eta)
\]
\[ z = \xi - \phi(\eta) \]

\[
\dot{\eta} = \left[ f_a(\eta) + g_a(\eta)\phi(\eta) \right] + g_a(\eta)z \\
\dot{z} = F(\eta, \xi) + g_b(\eta, \xi)u
\]

\[ V(\eta, \xi) = V_a(\eta) + \frac{1}{2}z^2 = V_a(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^2 \]

\[
\dot{V} = \frac{\partial V_a}{\partial \eta} \left[ f_a(\eta) + g_a(\eta)\phi(\eta) \right] + \frac{\partial V_a}{\partial \eta} g_a(\eta)z \\
+ zF(\eta, \xi) + zg_b(\eta, \xi)u \\
\leq -W(\eta) + z \left[ \frac{\partial V_a}{\partial \eta} g_a(\eta) + F(\eta, \xi) + g_b(\eta, \xi)u \right]
\]
\[
\dot{V} \leq -W(\eta) + z \left[ \frac{\partial V_a}{\partial \eta} g_a(\eta) + F(\eta, \xi) + g_b(\eta, \xi)u \right]
\]

\[
u = - \frac{1}{g_b(\eta, \xi)} \left[ \frac{\partial V_a}{\partial \eta} g_a(\eta) + F(\eta, \xi) + k z \right], \quad k > 0
\]

\[
\dot{V} \leq -W(\eta) - k z^2
\]
Example 9.9

\[ \dot{x}_1 = x_1^2 - x_1^3 + x_2, \quad \dot{x}_2 = u \]

\[ \dot{x}_1 = x_1^2 - x_1^3 + x_2 \]

\[ x_2 = \phi(x_1) = -x_1^2 - x_1 \quad \Rightarrow \quad \dot{x}_1 = -x_1 - x_1^3 \]

\[ V_a(x_1) = \frac{1}{2}x_1^2 \quad \Rightarrow \quad \dot{V}_a = -x_1^2 - x_1^4, \quad \forall \ x_1 \in R \]

\[ z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2 \]

\[ \dot{x}_1 = -x_1 - x_1^3 + z_2 \]

\[ \dot{z}_2 = u + (1 + 2x_1)(-x_1 - x_1^3 + z_2) \]
\( V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} z_2^2 \)

\[
\begin{align*}
\dot{V} &= x_1(-x_1 - x_1^3 + z_2) \\
&\quad + z_2[u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)]
\end{align*}
\]

\[
\begin{align*}
\dot{V} &= -x_1^2 - x_1^4 \\
&\quad + z_2[x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u]
\end{align*}
\]

\[ u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \]

\[
\dot{V} = -x_1^2 - x_1^4 - z_2^2
\]

The origin is globally asymptotically stable
Example 9.10

\[
\begin{align*}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u \\
\dot{x}_1 &= x_1^2 - x_1^3 + x_2, \quad \dot{x}_2 = x_3 \\
x_3 &= -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \overset{\text{def}}{=} \phi(x_1, x_2) \\
V_a(x) &= \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2, \quad \dot{V}_a = -x_1^2 - x_1^4 - z_2^2 \\
z_3 &= x_3 - \phi(x_1, x_2) \\
\dot{x}_1 &= x_1^2 - x_1^3 + x_2, \quad \dot{x}_2 = \phi(x_1, x_2) + z_3 \\
\dot{z}_3 &= u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3)
\end{align*}
\]
\[ V = V_a + \frac{1}{2}z_3^2 \]

\[ \dot{V} = \frac{\partial V_a}{\partial x_1} (x_1^2 - x_1^3 + x_2) + \frac{\partial V_a}{\partial x_2} (z_3 + \phi) + z_3 \left[ u - \frac{\partial \phi}{\partial x_1} (x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2} (z_3 + \phi) \right] \]

\[ V = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 \]

\[ + z_3 \left[ \frac{\partial V_a}{\partial x_2} - \frac{\partial \phi}{\partial x_1} (x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2} (z_3 + \phi) + u \right] \]

\[ u = -\frac{\partial V_a}{\partial x_2} + \frac{\partial \phi}{\partial x_1} (x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2} (z_3 + \phi) - z_3 \]

The origin is globally asymptotically stable
Strict-Feedback Form

\[
\begin{align*}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1})z_k \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k)u
\end{align*}
\]

\[g_i(x, z_1, \ldots, z_i) \neq 0 \quad \text{for } 1 \leq i \leq k\]