\[ \dot{x}_i = f_i(x_i, e_i), \quad y_i = h_i(x_i, e_i) \]

\[ y_i = h_i(t, e_i) \]
Passivity Theorems

Theorem 7.1
The feedback connection of two passive systems is passive

Proof
Let $V_1(x_1)$ and $V_2(x_2)$ be the storage functions for $H_1$ and $H_2$ ($V_i = 0$ if $H_i$ is memoryless)

$$e_i^T y_i \geq \dot{V}_i, \quad V(x) = V_1(x_1) + V_2(x_2)$$

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$u^T y = u_1^T y_1 + u_2^T y_2 \geq \dot{V}_1 + \dot{V}_2 = \dot{V}$$
Asymptotic Stability

**Theorem 7.2**

Consider the feedback connection of two dynamical systems. When \( u = 0 \), the origin of the closed-loop system is asymptotically stable if one of the following conditions is satisfied:

- both feedback components are strictly passive;
- both feedback components are output strictly passive and zero-state observable;
- one component is strictly passive and the other one is output strictly passive and zero-state observable.

If the storage function for each component is radially unbounded, the origin is globally asymptotically stable.
**Proof**

$H_1$ is SP; $H_2$ is OSP & ZSO

\[ e_1^T y_1 \geq \dot{V}_1 + \psi_1(x_1), \quad \psi_1(x_1) > 0, \quad \forall \ x_1 \neq 0 \]

\[ e_2^T y_2 \geq \dot{V}_2 + y_2^T \rho_2(y_2), \quad y_2^T \rho(y_2) > 0, \quad \forall y_2 \neq 0 \]

\[ e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2 \]

\[ V(x) = V_1(x_1) + V_2(x_2) \]

\[ \dot{V} \leq u^T y - \psi_1(x_1) - y_2^T \rho_2(y_2) \]

\[ u = 0 \quad \Rightarrow \quad \dot{V} \leq -\psi_1(x_1) - y_2^T \rho_2(y_2) \]
\[
\dot{V} \leq -\psi_1(x_1) - y_2^T \rho_2(y_2)
\]

\[
\dot{V} = 0 \quad \Rightarrow \quad x_1 = 0 \text{ and } y_2 = 0
\]

\[
y_2(t) \equiv 0 \quad \Rightarrow \quad e_1(t) \equiv 0 \quad (\& \quad x_1(t) \equiv 0) \quad \Rightarrow \quad y_1(t) \equiv 0
\]

\[
y_1(t) \equiv 0 \quad \Rightarrow \quad e_2(t) \equiv 0
\]

By zero-state observability of \(H_2\): \(y_2(t) \equiv 0 \quad \Rightarrow \quad x_2(t) \equiv 0\)

Apply the invariance principle.
Example 7.1

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -ax_1^3 - kx_2 + e_1 \\
y_1 &= x_2 
\end{align*}
\]

\[
\begin{align*}
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -bx_3 - x_4^3 + e_2 \\
y_2 &= x_4 
\end{align*}
\]

\( H_1 \)

\( a, b, k > 0 \)

\[
V_1 = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2 
\]

\[
\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - kx_2^2 + x_2e_1 = -ky_1^2 + y_1e_1 
\]

With \( e_1 = 0 \), \( y_1(t) \equiv 0 \ \Leftrightarrow \ x_2(t) \equiv 0 \ \Rightarrow \ x_1(t) \equiv 0 \)

\( H_1 \) is output strictly passive and zero-state observable
$V_2 = \frac{1}{2} b x_3^2 + \frac{1}{2} x_4^2$

$\dot{V}_2 = b x_3 x_4 - b x_3 x_4 - x_4^4 + x_4 e_2 = -y_2^4 + y_2 e_2$

With $e_2 = 0$, $y_2(t) \equiv 0 \iff x_4(t) \equiv 0 \Rightarrow x_3(t) \equiv 0$

$H_2$ is output strictly passive and zero-state observable

$V_1$ and $V_2$ are radially unbounded

The origin is globally asymptotically stable
Theorem 7.3

Consider the feedback connection of a strictly passive dynamical system with a passive time-varying memoryless function. When \( u = 0 \), the origin of the closed-loop system is uniformly asymptotically stable. If the storage function for the dynamical system is radially unbounded, the origin will be globally uniformly asymptotically stable.

Proof

Let \( V_1(x_1) \) be (positive definite) storage function of \( H_1 \).

\[
\dot{V}_1 = \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) \leq e_1^T y_1 - \psi_1(x_1) = -e_2^T y_2 - \psi_1(x_1)
\]

\[
e_2^T y_2 \geq 0 \quad \Rightarrow \quad \dot{V}_1 \leq -\psi_1(x_1)
\]
Example 7.4

Consider the feedback connection of a strictly positive real transfer function and a passive time-varying memoryless function

From Lemma 5.4, we know that the dynamical system is strictly passive with a positive definite storage function

\[ V(x) = \frac{1}{2} x^T P x \]

From Theorem 7.3, the origin of the closed-loop system is globally uniformly asymptotically stable
Theorem 7.4

Consider the feedback connection of a time-invariant dynamical system $H_1$ with a time-invariant memoryless function $H_2$. Suppose $H_1$ is zero-state observable, $V_1(x_1)$ is positive definite

$$e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1), \quad e_2^T y_2 \geq e_2^T \varphi_2(e_2)$$

Then, the origin of the closed-loop system (when $u = 0$) is asymptotically stable if

$$v^T [\rho_1(v) + \varphi_2(v)] > 0, \quad \forall \ v \neq 0$$

Furthermore, if $V_1$ is radially unbounded, the origin will be globally asymptotically stable
Example 7.5

\[
\begin{align*}
\dot{x} &= f(x) + G(x)e_1 \\
y_1 &= h(x) \\
\end{align*}
\]

Suppose \( H_1 \) is zero-state observable and there is a radially unbounded positive definite function \( V_1(x) \) such that

\[
\frac{\partial V_1}{\partial x} f(x) \leq 0, \quad \frac{\partial V_1}{\partial x} G(x) = h^T(x), \quad \forall \ x \in R^n
\]

\[
\dot{V}_1 = \frac{\partial V_1}{\partial x} f(x) + \frac{\partial V_1}{\partial x} G(x)e_1 \leq y_1^T e_1
\]
Apply Theorem 7.4:

\[ \dot{V}_1 \leq e_1^T y_1 \]

\[ e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1) \quad \text{is satisfied with} \quad \rho_1 = 0 \]

\[ e_2^T y_2 = e_2^T \sigma(e_2) \]

\[ e_2^T y_2 \geq e_2^T \varphi_2(e_2) \quad \text{is satisfied with} \quad \varphi_2 = \sigma \]

\[ v^T [\rho_1(v) + \varphi_2(v)] = v^T \sigma(v) > 0, \quad \forall \ v \neq 0 \]

The origin is globally asymptotically stable
The Small-Gain Theorem

\[ \|y_1\| \leq \gamma_1 \|e_1\| \beta_1, \quad \forall \ e_1 \in \mathcal{L}_e^m, \forall \ \tau \in [0, \infty) \]

\[ \|y_2\| \leq \gamma_2 \|e_2\| \beta_2, \quad \forall \ e_2 \in \mathcal{L}_e^q, \forall \ \tau \in [0, \infty) \]
\[ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \]

**Theorem 7.7**

The feedback connection is finite-gain \( \mathcal{L} \) stable if \( \gamma_1 \gamma_2 < 1 \)
Definition 7.1

The system is absolutely stable if the origin is globally uniformly asymptotically stable for any nonlinearity in a given sector. It is absolutely stable with finite domain if the origin is uniformly asymptotically stable.
Circle Criterion

Suppose $G(s) = C(sI - A)^{-1}B + D$ is SPR, $\psi \in [0, \infty]$

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
u &= -\psi(t, y)
\end{align*}
\]

By the KYP Lemma, $\exists P = P^T > 0$, $L, W, \varepsilon > 0$

\[
\begin{align*}
PA + A^TP &= -L^TL - \varepsilon P \\
PB &= C^T - L^TW \\
W^TW &= D + D^T
\end{align*}
\]

\[V(x) = \frac{1}{2}x^TPx\]
\[ \dot{V} = \frac{1}{2} x^T P \dot{x} + \frac{1}{2} \dot{x}^T P x \]
\[ = \frac{1}{2} x^T (PA + A^T P)x + x^T PBu \]
\[ = - \frac{1}{2} x^T L^T Lx - \frac{1}{2} \varepsilon x^T P x + x^T (C^T - L^T W)u \]
\[ = - \frac{1}{2} x^T L^T Lx - \frac{1}{2} \varepsilon x^T P x + (Cx + Du)^T u \]
\[ - u^T Du - x^T L^T W u \]
\[ u^T Du = \frac{1}{2} u^T (D + D^T)u = \frac{1}{2} u^T W^T W u \]
\[ \dot{V} = - \frac{1}{2} \varepsilon x^T P x - \frac{1}{2} (Lx + Wu)^T (Lx + Wu) - y^T \psi(t, y) \]
\[ y^T \psi(t, y) \geq 0 \quad \Rightarrow \quad \dot{V} \leq - \frac{1}{2} \varepsilon x^T P x \]

The origin is globally exponentially stable
What if $\psi \in [K_1, \infty]$?

$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $G(s)[I + K_1 G(s)]^{-1}$ is SPR
What if $\psi \in [K_1, K_2]$?

$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $I + KG(s)[I + K_1 G(s)]^{-1}$ is SPR.
\[ I + KG(s)[I + K_1G(s)]^{-1} = [I + K_2G(s)][I + K_1G(s)]^{-1} \]

**Theorem 7.8 (Circle Criterion)**

The system is absolutely stable if
- \( \psi \in [K_1, \infty] \) and \( G(s)[I + K_1G(s)]^{-1} \) is SPR, or
- \( \psi \in [K_1, K_2] \) and \( [I + K_2G(s)][I + K_1G(s)]^{-1} \) is SPR

If the sector condition is satisfied only on a set \( Y \subset R^m \), then the foregoing conditions ensure absolute stability with finite domain
Scalar Case: \( \psi \in [\alpha, \beta], \ \beta > \alpha \)

The system is absolutely stable if

\[
\frac{1 + \beta G(s)}{1 + \alpha G(s)} \text{ is Hurwitz and }
\]

\[
\text{Re} \left[ \frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)} \right] > 0, \ \forall \ \omega \in [0, \infty]
\]
Case 1:  $\alpha > 0$

By the Nyquist criterion

$$\frac{1 + \beta G(s)}{1 + \alpha G(s)} = \frac{1}{1 + \alpha G(s)} + \frac{\beta G(s)}{1 + \alpha G(s)}$$

is Hurwitz if the Nyquist plot of $G(j\omega)$ does not intersect the point $-(1/\alpha) + j0$ and encircles it $p$ times in the counterclockwise direction, where $p$ is the number of poles of $G(s)$ in the open right-half complex plane

$$\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)} > 0 \iff \frac{1}{\beta} + G(j\omega) > 0 \quad \frac{1}{\alpha} + G(j\omega) > 0$$
The system is absolutely stable if the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha, \beta)$ and encircles it $p$ times in the counterclockwise direction.
Theorem 7.9

Consider an SISO \( G(s) \) and \( \psi \in [\alpha, \beta] \). Then, the system is absolutely stable if one of the following conditions is satisfied.

1. \( 0 < \alpha < \beta \), the Nyquist plot of \( G(s) \) does not enter the disk \( D(\alpha, \beta) \) and encircles it \( p \) times in the counterclockwise direction, where \( p \) is the number of poles of \( G(s) \) with positive real parts.

2. \( 0 = \alpha < \beta \), \( G(s) \) is Hurwitz and the Nyquist plot of \( G(s) \) lies to the right of the vertical line \( \text{Re}[s] = -1/\beta \).

3. \( \alpha < 0 < \beta \), \( G(s) \) is Hurwitz and the Nyquist plot of \( G(s) \) lies in the interior of the disk \( D(\alpha, \beta) \).

If the sector condition is satisfied only on an interval \([a, b]\), then the foregoing conditions ensure absolute stability with finite domain.
Popov Criterion

\[ \dot{x} = Ax + Bu, \quad y = Cx \]

\((A, B)\) controllable, \((A, C)\) observable

\[ u_i = -\psi_i(y_i), \quad \psi_i \in [0, k_i], \quad 1 \leq i \leq m, \quad (0 < k_i \leq \infty) \]

\[ G(s) = C(sI - A)^{-1}B \]

\[ \Gamma = \text{diag}(\gamma_1, \ldots, \gamma_m), \quad M = \text{diag}(1/k_1, \ldots, 1/k_m) \]
The system is absolutely stable if for $1 \leq i \leq m$,

$$
\psi_i \in [0, k_i], \quad 0 < k_i \leq \infty
$$

and there is $\gamma_i \geq 0$, with $(1 + \lambda_k \gamma_i) \neq 0$ for every eigenvalue $\lambda_k$ of $A$, such that

$$
M + (I + s\Gamma)G(s) \text{ is SPR}
$$

If the sector condition $\psi_i \in [0, k_i]$ is satisfied only on a set $Y \subset R^m$, then the system is absolutely stable with finite domain.
Scalar case

\[ \frac{1}{k} + (1 + s\gamma)G(s) \]

is SPR if \( G(s) \) is Hurwitz and

\[ \frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega \text{Im}[G(j\omega)] > 0, \quad \forall \omega \in [0, \infty) \]

If

\[ \lim_{\omega \to \infty} \left\{ \frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega \text{Im}[G(j\omega)] \right\} = 0 \]

we also need

\[ \lim_{\omega \to \infty} \omega^2 \left\{ \frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega \text{Im}[G(j\omega)] \right\} > 0 \]
\[
\frac{1}{k} + \text{Re}[G(j\omega)] - \gamma \omega \text{Im}[G(j\omega)] > 0, \quad \forall \omega \in [0, \infty)
\]

Popov Plot
Example

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 - h(y), \quad y = x_1 \]

\[ \dot{x}_2 = -\alpha x_1 - x_2 - h(y) + \alpha x_1, \quad \alpha > 0 \]

\[ G(s) = \frac{1}{s^2 + s + \alpha}, \quad \psi(y) = h(y) - \alpha y \]

\[ h \in [\alpha, \beta] \implies \psi \in [0, k] \quad (k = \beta - \alpha > 0) \]

\[ \gamma > 1 \implies \frac{\alpha - \omega^2 + \gamma \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall \omega \in [0, \infty) \]

\[ \text{and} \quad \lim_{\omega \to \infty} \frac{\omega^2(\alpha - \omega^2 + \gamma \omega^2)}{(\alpha - \omega^2)^2 + \omega^2} = \gamma - 1 > 0 \]
The system is absolutely stable for $\psi \in [0, \infty)$ ($h \in [\alpha, \infty]$)

Compare with the circle criterion ($\gamma = 0$)

$$\frac{1}{k} + \frac{\alpha - \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall \omega \in [0, \infty], \quad \text{for } k < 1 + 2\sqrt{\alpha}$$