Nonlinear Control

Lecture # 6

Passivity

and

Input-Output Stability
Passivity: Memoryless Functions

Passive

Passive

Not passive

\[ y = h(t, u), \quad h \in [0, \infty] \]

Vector case:

\[ y = h(t, u), \quad h^T = [h_1, h_2, \cdots, h_p] \]

Power inflow:

\[ \text{power inflow} = \sum_{i=1}^{p} u_i y_i = u^T y \]
Definition 5.1

\( y = h(t, u) \) is

- passive if \( u^T y \geq 0 \)
- lossless if \( u^T y = 0 \)
- input strictly passive if \( u^T y \geq u^T \varphi(u) \) for some function \( \varphi \) where \( u^T \varphi(u) > 0 \), \( \forall u \neq 0 \)
- output strictly passive if \( u^T y \geq y^T \rho(y) \) for some function \( \rho \) where \( y^T \rho(y) > 0 \), \( \forall y \neq 0 \)
Sector Nonlinearity: \( h \) belongs to the sector \([\alpha, \beta]\) \((h \in [\alpha, \beta])\) if

\[ \alpha u^2 \leq uh(t, u) \leq \beta u^2 \]

Also,

\[ h \in (\alpha, \beta], \quad h \in [\alpha, \beta), \quad h \in (\alpha, \beta) \]
\[ \alpha u^2 \leq uh(t, u) \leq \beta u^2 \iff [h(t, u) - \alpha u][h(t, u) - \beta u] \leq 0 \]

**Definition 5.2**

A memoryless function \( h(t, u) \) is said to belong to the sector

- \([0, \infty]\) if \( u^T h(t, u) \geq 0 \)
- \([K_1, \infty]\) if \( u^T [h(t, u) - K_1 u] \geq 0 \)
- \([0, K_2]\) with \( K_2 = K_2^T > 0 \) if \( h^T(t, u)[h(t, u) - K_2 u] \leq 0 \)
- \([K_1, K_2]\) with \( K = K_2 - K_1 = K^T > 0 \) if

\[ [h(t, u) - K_1 u]^T [h(t, u) - K_2 u] \leq 0 \]
A function in the sector \([K_1, K_2]\) can be transformed into a function in the sector \([0, \infty]\) by input feedforward followed by output feedback.
Passivity: State Models

Definition 5.3
The system
\[
\dot{x} = f(x, u), \quad y = h(x, u)
\]
is passive if there is a continuously differentiable positive semidefinite function \( V(x) \) (the storage function) such that
\[
u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u), \quad \forall (x, u)\]
Moreover, it is
- lossless if \( u^T y = \dot{V} \)
- input strictly passive if \( u^T y \geq \dot{V} + u^T \varphi(u) \) for some function \( \varphi \) such that \( u^T \varphi(u) > 0, \quad \forall u \neq 0 \)
- output strictly passive if \( u^T y \geq \dot{V} + y^T \rho(y) \) for some function \( \rho \) such that \( y^T \rho(y) > 0, \quad \forall y \neq 0 \)
- strictly passive if \( u^T y \geq \dot{V} + \psi(x) \) for some positive definite function \( \psi \)
Example 5.2

\[ \dot{x} = u, \quad y = x \]

\[ V(x) = \frac{1}{2}x^2 \quad \Rightarrow \quad uy = \dot{V} \quad \Rightarrow \quad \text{Lossless} \]

\[ \dot{x} = u, \quad y = x + h(u), \quad h \in [0, \infty] \]

\[ V(x) = \frac{1}{2}x^2 \quad \Rightarrow \quad uy = \dot{V} + uh(u) \quad \Rightarrow \quad \text{Passive} \]

\[ h \in (0, \infty] \quad \Rightarrow \quad uh(u) > 0 \quad \forall \quad u \neq 0 \]

\[ \Rightarrow \quad \text{Input strictly passive} \]

\[ \dot{x} = -h(x) + u, \quad y = x, \quad h \in [0, \infty] \]

\[ V(x) = \frac{1}{2}x^2 \quad \Rightarrow \quad uy = \dot{V} + yh(y) \quad \Rightarrow \quad \text{Passive} \]

\[ h \in (0, \infty] \quad \Rightarrow \quad \text{Output strictly passive} \]
Example 5.3

\[ \dot{x} = u, \quad y = h(x), \quad h \in [0, \infty] \]

\[ V(x) = \int_0^x h(\sigma) \, d\sigma \quad \Rightarrow \quad \dot{V} = h(x)\dot{x} = yu \quad \Rightarrow \quad \text{Lossless} \]

\[ a\dot{x} = -x + u, \quad y = h(x), \quad h \in [0, \infty] \]

\[ V(x) = a \int_0^x h(\sigma) \, d\sigma \quad \Rightarrow \quad \dot{V} = h(x)(-x + u) = yu - xh(x) \]

\[ yu = \dot{V} + xh(x) \quad \Rightarrow \quad \text{Passive} \]

\[ h \in (0, \infty] \quad \Rightarrow \quad \text{Strictly passive} \]
Example 5.4

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -h(x_1) - ax_2 + u, \\
y &= bx_2 + u
\end{align*}
\]

\[h \in [\alpha_1, \infty], \quad a > 0, \quad b > 0, \quad \alpha_1 > 0\]

\[
V(x) = \alpha \int_0^{x_1} h(\sigma) \, d\sigma + \frac{1}{2} \alpha x^T P x
\]

\[
= \alpha \int_0^{x_1} h(\sigma) \, d\sigma + \frac{1}{2} \alpha (p_{11} x_1^2 + 2p_{12} x_1 x_2 + p_{22} x_2^2)
\]

\[\alpha > 0, \quad p_{11} > 0, \quad p_{11} p_{22} - p_{12}^2 > 0\]
\[ \dot{V} = u(bx_2 + u) - \alpha[h(x_1) + p_{11}x_1 + p_{12}x_2]x_2 \]
\[ - \alpha(p_{12}x_1 + p_{22}x_2)[-h(x_1) - ax_2 + u] \]

Take \( p_{22} = 1, \ p_{11} = ap_{12}, \) and \( \alpha = b \) to cancel the cross product terms

\[ \dot{V} \geq bp_{12} \left( \alpha_1 - \frac{1}{4}bp_{12} \right)x_1^2 + b(a - p_{12})x_2^2 \]

\[ p_{12} = ak, \quad 0 < k < \min\{1, 4\alpha_1/(ab)\} \]

\[ \Rightarrow p_{11} > 0, \ p_{11}p_{22} - p_{12}^2 > 0 \]

\[ \Rightarrow \quad \text{Strictly passive} \]
Definition 5.4

An $m \times m$ proper rational transfer function matrix $G(s)$ is positive real if

- poles of all elements of $G(s)$ are in $\text{Re}[s] \leq 0$
- for all real $\omega$ for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite
- any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian

$G(s)$ is strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$
Scalar Case ($m = 1$):

$$G(j\omega) + G^T(-j\omega) = 2\text{Re}[G(j\omega)]$$

$\text{Re}[G(j\omega)]$ is an even function of $\omega$. The second condition of the definition reduces to

$$\text{Re}[G(j\omega)] \geq 0, \forall \omega \in [0, \infty)$$

which holds when the Nyquist plot of $G(j\omega)$ lies in the closed right-half complex plane.

This is true only if the relative degree of the transfer function is zero or one.
Lemma 5.1

An $m \times m$ proper rational transfer function matrix $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $G(j\omega) + G^T(-j\omega) > 0$, $\forall \omega \in \mathbb{R}$
- $G(\infty) + G^T(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^{2(m-q)} \det [G(j\omega) + G^T(-j\omega)] > 0$$

where $q = \text{rank}[G(\infty) + G^T(\infty)]$
Scalar Case \((m = 1)\): \(G(s)\) is strictly positive real if and only if

- \(G(s)\) is Hurwitz
- \(\text{Re}[G(j\omega)] > 0, \quad \forall \omega \in [0, \infty)\)
- \(G(\infty) > 0\) or

\[
\lim_{\omega \to \infty} \omega^2 \text{Re}[G(j\omega)] > 0
\]
Positive Real Lemma (5.2)

Let

\[ G(s) = C(sI - A)^{-1}B + D \]

where \((A, B)\) is controllable and \((A, C)\) is observable. \(G(s)\) is positive real if and only if there exist matrices \(P = P^T > 0\), \(L\), and \(W\) such that

\[
\begin{align*}
PA + A^TP &= -L^TL \\
PB &= CT - LTW \\
WTW &= D + DT
\end{align*}
\]
Kalman–Yakubovich–Popov Lemma (5.3)

Let

\[ G(s) = C(sI - A)^{-1}B + D \]

where \((A, B)\) is controllable and \((A, C)\) is observable. \(G(s)\) is strictly positive real if and only if there exist matrices \(P = P^T > 0\), \(L\), and \(W\), and a positive constant \(\varepsilon\) such that

\[
\begin{align*}
PA + A^TP &= -L^TL - \varepsilon P \\
PB &= C^T - L^TW \\
W^TW &= D + D^T
\end{align*}
\]
Lemma 5.4

The linear time-invariant minimal realization

\[ \dot{x} = Ax + Bu, \quad y = Cx + Du \]

with

\[ G(s) = C(sI - A)^{-1}B + D \]

is

- passive if \( G(s) \) is positive real
- strictly passive if \( G(s) \) is strictly positive real

Proof

Apply the PR and KYP Lemmas, respectively, and use

\[ V(x) = \frac{1}{2}x^T Px \] as the storage function
Connection with Stability

Lemma 5.5
If the system
\[ \dot{x} = f(x, u), \quad y = h(x, u) \]
is passive with a positive definite storage function \( V(x) \), then the origin of \( \dot{x} = f(x, 0) \) is stable.

Proof
\[ u^T y \geq \frac{\partial V}{\partial x} f(x, u) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, 0) \leq 0 \]
Lemma 5.6

If the system

\[ \dot{x} = f(x, u), \quad y = h(x, u) \]

is strictly passive, then the origin of \( \dot{x} = f(x, 0) \) is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable.

Proof

The storage function \( V(x) \) is positive definite

\[ u^T y \geq \frac{\partial V}{\partial x} f(x, u) + \psi(x) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, 0) \leq -\psi(x) \]
Definition 5.5

The system

\[ \dot{x} = f(x, u), \quad y = h(x, u) \]

is zero-state observable if no solution of \( \dot{x} = f(x, 0) \) can stay identically in \( S = \{ h(x, 0) = 0 \} \), other than the zero solution \( x(t) \equiv 0 \).

Linear Systems

\[ \dot{x} = Ax, \quad y = Cx \]

Observability of \((A, C)\) is equivalent to

\[ y(t) = Ce^{At}x(0) \equiv 0 \iff x(0) = 0 \iff x(t) \equiv 0 \]
Lemma 5.6

If the system
\[ \dot{x} = f(x, u), \quad y = h(x, u) \]
is output strictly passive and zero-state observable, then the origin of \( \dot{x} = f(x, 0) \) is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable.

Proof

The storage function \( V(x) \) is positive definite

\[ u^T y \geq \frac{\partial V}{\partial x} f(x, u) + y^T \rho(y) \Rightarrow \frac{\partial V}{\partial x} f(x, 0) \leq -y^T \rho(y) \]

\[ \dot{V}(x(t)) \equiv 0 \Rightarrow y(t) \equiv 0 \Rightarrow x(t) \equiv 0 \]

Apply the invariance principle.
Example 5.8

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_1^3 - kx_2 + u, \quad y = x_2, \quad a, k > 0 \]

\[ V(x) = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2 \]

\[ \dot{V} = ax_1^3x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu \]

The system is output strictly passive

\[ y(t) \equiv 0 \iff x_2(t) \equiv 0 \implies ax_1^3(t) \equiv 0 \implies x_1(t) \equiv 0 \]

The system is zero-state observable. \( V \) is radially unbounded. Hence, the origin of the unforced system is globally asymptotically stable
\[ \mathcal{L} \text{ Stability} \]

**Input-Output Models:** \( y = H u \)

\( u(t) \) is a piecewise continuous function of \( t \) and belongs to a linear space of signals

- The space of bounded functions: \( \sup_{t \geq 0} \| u(t) \| < \infty \)
- The space of square-integrable functions:
  \[ \int_0^\infty u^T(t)u(t) \, dt < \infty \]

Norm of a signal \( \| u \| \):

- \( \| u \| \geq 0 \) and \( \| u \| = 0 \iff u = 0 \)
- \( \| au \| = a \| u \| \text{ for any } a > 0 \)
- Triangle Inequality: \( \| u_1 + u_2 \| \leq \| u_1 \| + \| u_2 \| \)
\( \mathcal{L}_p \) spaces:

\[ \mathcal{L}_\infty : \quad \|u\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \|u(t)\| < \infty \]

\[ \mathcal{L}_2 : \quad \|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t) \, dt} < \infty \]

\[ \mathcal{L}_p : \quad \|u\|_{\mathcal{L}_p} = \left( \int_0^\infty \|u(t)\|^p \, dt \right)^{1/p} < \infty, \quad 1 \leq p < \infty \]

**Notation** \( \mathcal{L}^m_p \): \( p \) is the type of \( p \)-norm used to define the space and \( m \) is the dimension of \( u \)
Extended Space: \( L_e = \{u \mid u_\tau \in L, \forall \tau \in [0, \infty)\} \)

\( u_\tau \) is a truncation of \( u \): \( u_\tau(t) = \begin{cases} u(t), & 0 \leq t \leq \tau \\ 0, & t > \tau \end{cases} \)

\( L_e \) is a linear space and \( L \subset L_e \)

Example

\[ u(t) = t, \quad u_\tau(t) = \begin{cases} t, & 0 \leq t \leq \tau \\ 0, & t > \tau \end{cases} \]

\( u \notin L_\infty \) but \( u_\tau \in L_\infty^e \)
**Causality:** A mapping $H: \mathcal{L}_e^m \to \mathcal{L}_e^n$ is causal if the value of the output $(Hu)(t)$ at any time $t$ depends only on the values of the input up to time $t$

$$(Hu)_\tau = (Hu_\tau)_\tau$$

**Definition 6.1**

A scalar continuous function $g(r)$, defined for $r \in [0, a)$, is a gain function if it is nondecreasing and $g(0) = 0$

A class $\mathcal{K}$ function is a gain function but not the other way around. By not requiring the gain function to be strictly increasing we can have $g = 0$ or $g(r) = \text{sat}(r)$
Definition 6.2

A mapping $H : \mathcal{L}_e^m \rightarrow \mathcal{L}_e^q$ is $\mathcal{L}$ stable if there exist a gain function $g$, defined on $[0, \infty)$, and a nonnegative constant $\beta$ such that

$$\| (Hu)_\tau \|_\mathcal{L} \leq g(\| u_\tau \|_\mathcal{L}) + \beta, \quad \forall \ u \in \mathcal{L}_e^m \text{ and } \tau \in [0, \infty)$$

It is finite-gain $\mathcal{L}$ stable if there exist nonnegative constants $\gamma$ and $\beta$ such that

$$\| (Hu)_\tau \|_\mathcal{L} \leq \gamma \| u_\tau \|_\mathcal{L} + \beta, \quad \forall \ u \in \mathcal{L}_e^m \text{ and } \tau \in [0, \infty)$$

In this case, we say that the system has $\mathcal{L}$ gain $\leq \gamma$. The bias term $\beta$ is included in the definition to allow for systems where $Hu$ does not vanish at $u = 0$. 
\( \dot{x} = f(x, u), \quad y = h(x, u), \quad 0 = f(0, 0), \quad 0 = h(0, 0) \)

**Case 1:** The origin of \( \dot{x} = f(x, 0) \) is exponentially stable

**Theorem 6.1**

Suppose, \( \forall \|x\| \leq r, \forall \|u\| \leq r_u, \)

\[
c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2
\]

\[
\frac{\partial V}{\partial x} f(x, 0) \leq -c_3\|x\|^2, \quad \left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x\|
\]

\[
\|f(x, u) - f(x, 0)\| \leq L\|u\|, \quad \|h(x, u)\| \leq \eta_1\|x\| + \eta_2\|u\|
\]

Then, for each \( x_0 \) with \( \|x_0\| \leq r \sqrt{c_1/c_2} \), the system is small-signal finite-gain \( \mathcal{L}_p \) stable for each \( p \in [1, \infty] \). It is finite-gain \( \mathcal{L}_p \) stable \( \forall \ x_0 \in \mathbb{R}^n \) if the assumptions hold globally [see the textbook for \( \beta \) and \( \gamma \)]
Example 6.4

\[ \dot{x} = -x - x^3 + u, \quad y = \tanh x + u \]

\[ V = \frac{1}{2}x^2 \quad \Rightarrow \quad \dot{V} = x(-x - x^3) \leq -x^2 \]

\[ c_1 = c_2 = \frac{1}{2}, \quad c_3 = c_4 = 1, \quad L = \eta_1 = \eta_2 = 1 \]

Finite-gain \( \mathcal{L}_p \) stable for each \( x(0) \in \mathbb{R} \) and each \( p \in [1, \infty] \)

Example 6.5

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 - a \tanh x_1 + u, \quad y = x_1, \quad a \geq 0 \]

\[ V(x) = x^T Px = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2 \]
\[
\dot{V} = -2p_{12}(x_1^2 + ax_1 \tanh x_1) + 2(p_{11} - p_{12} - p_{22})x_1x_2
- 2ap_{22}x_2 \tanh x_1 - 2(p_{22} - p_{12})x_2^2
\]

\[
p_{11} = p_{12} + p_{22} \quad \Rightarrow \quad \text{the term } x_1x_2 \text{ is canceled}
\]

\[
p_{22} = 2p_{12} = 1 \quad \Rightarrow \quad P \text{ is positive definite}
\]

\[
\dot{V} = -x_1^2 - x_2^2 - ax_1 \tanh x_1 - 2ax_2 \tanh x_1
\]

\[
\dot{V} \leq -\|x\|^2 + 2a|x_1| |x_2| \leq -(1 - a)\|x\|^2
\]

\[
a < 1 \Rightarrow c_1 = \lambda_{\min}(P), \quad c_2 = \lambda_{\max}(P), \quad c_3 = 1 - a, \quad c_4 = 2c_2
\]

\[
L = \eta_1 = 1, \quad \eta_2 = 0
\]

For each \(x(0) \in R^2, \ p \in [1, \infty]\), the system is finite-gain \(L_p\) stable

\[
\gamma = 2[\lambda_{\max}(P)]^2/[(1 - a)\lambda_{\min}(P)]
\]
Case 2: The origin of $\dot{x} = f(x, 0)$ is asymptotically stable

**Theorem 6.2**

Suppose that, for all $(x, u)$, $f$ is locally Lipschitz and $h$ is continuous and satisfies

$$\|h(x, u)\| \leq g_1(\|x\|) + g_2(\|u\|) + \eta, \quad \eta \geq 0$$

for some gain functions $g_1, g_2$. If $\dot{x} = f(x, u)$ is ISS, then, for each $x(0) \in R^n$, the system

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

is $L_\infty$ stable
Example 6.6

\[
\dot{x} = -x - 2x^3 + (1 + x^2)u^2, \quad y = x^2 + u
\]

ISS from Example 4.13

\[
g_1(r) = r^2, \quad g_2(r) = r, \quad \eta = 0
\]

\[ \mathcal{L}_\infty \text{ stable} \]
Example 6.7

\[
\dot{x}_1 = -x_1^3 + x_2, \quad \dot{x}_2 = -x_1 - x_2^3 + u, \quad y = x_1 + x_2
\]

\[
V = (x_1^2 + x_2^2) \quad \Rightarrow \quad \dot{V} = -2x_1^4 - 2x_2^4 + 2x_2u
\]

\[
x_1^4 + x_2^4 \geq \frac{1}{2} \|x\|^4
\]

\[
\dot{V} \leq -\|x\|^4 + 2\|x\||u|
\]

\[
= -(1 - \theta)\|x\|^4 - \theta\|x\|^4 + 2\|x\||u|, \quad 0 < \theta < 1
\]

\[
\leq -(1 - \theta)\|x\|^4, \quad \forall \|x\| \geq \left(\frac{2|u|}{\theta}\right)^{1/3} \Rightarrow \text{ISS}
\]

\[
g_1(r) = \sqrt{2}r, \quad g_2 = 0, \quad \eta = 0
\]

\[\mathcal{L}_\infty \text{ stable}\]