Nonlinear Control

Lecture # 8
Time Varying
and
Perturbed Systems
Time-varying Systems

\[ \dot{x} = f(t, x) \]

\( f(t, x) \) is piecewise continuous in \( t \) and locally Lipschitz in \( x \) for all \( t \geq 0 \) and all \( x \in D, (0 \in D) \). The origin is an equilibrium point at \( t = 0 \) if

\[ f(t, 0) = 0, \; \forall t \geq 0 \]

While the solution of the time-invariant system

\[ \dot{x} = f(x), \quad x(t_0) = x_0 \]

depends only on \( (t - t_0) \), the solution of

\[ \dot{x} = f(t, x), \quad x(t_0) = x_0 \]

may depend on both \( t \) and \( t_0 \)
Comparison Functions

- A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$, belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if it is defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.

- A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$, belongs to class $\mathcal{KL}$ if, for each fixed $s$, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to $r$ and, for each fixed $r$, the mapping $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to \infty$. 

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Nonlinear Control Lecture # 8 Time Varying and Perturbed Systems
Example 4.1

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1 + r^2) > 0$. It belongs to class $K$, but not to class $K_\infty$ since $\lim_{r \to \infty} \alpha(r) = \pi/2 < \infty$.

- $\alpha(r) = r^c$, $c > 0$, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r \to \infty} \alpha(r) = \infty$; thus, it belongs to class $K_\infty$.

- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r \to \infty} \alpha(r) = \infty$. Hence, it belongs to class $K_\infty$. It is not continuously differentiable at $r = 1$. Continuous differentiability is not required for a class $K$ function.
\( \beta(r, s) = r / (ksr + 1) \), for any positive constant \( k \), is strictly increasing in \( r \) since
\[
\frac{\partial \beta}{\partial r} = \frac{1}{(ksr + 1)^2} > 0
\]
and strictly decreasing in \( s \) since
\[
\frac{\partial \beta}{\partial s} = -\frac{kr^2}{(ksr + 1)^2} < 0
\]
\( \beta(r, s) \to 0 \) as \( s \to \infty \). It belongs to class \( \mathcal{KL} \)

\[ \beta(r, s) = r^c e^{-as} \], with positive constants \( a \) and \( c \), belongs to class \( \mathcal{KL} \)
Lemma 4.1

Let $\alpha_1$ and $\alpha_2$ be class $\mathcal{K}$ functions on $[0, a_1)$ and $[0, a_2)$, respectively, with $a_1 \geq \lim_{r \to a_2} \alpha_2(r)$, and $\beta$ be a class $\mathcal{KL}$ function defined on $[0, \lim_{r \to a_2} \alpha_2(r)) \times [0, \infty)$ with $a_1 \geq \lim_{r \to a_2} \beta(\alpha_2(r), 0)$. Let $\alpha_3$ and $\alpha_4$ be class $\mathcal{K}_\infty$ functions. Denote the inverse of $\alpha_i$ by $\alpha_i^{-1}$. Then,

- $\alpha_1^{-1}$ is defined on $[0, \lim_{r \to a_1} \alpha_1(r))$ and belongs to class $\mathcal{K}$
- $\alpha_3^{-1}$ is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$
- $\alpha_1 \circ \alpha_2$ is defined on $[0, a_2)$ and belongs to class $\mathcal{K}$
- $\alpha_3 \circ \alpha_4$ is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$
- $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is defined on $[0, a_2) \times [0, \infty)$ and belongs to class $\mathcal{KL}$
Lemma 4.2

Let $V : D \to \mathbb{R}$ be a continuous positive definite function defined on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions $\alpha_1$ and $\alpha_2$, defined on $[0, r]$, such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

for all $x \in B_r$. If $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then there exist class $\mathcal{K}_\infty$ functions $\alpha_1$ and $\alpha_2$ such that the foregoing inequality holds for all $x \in \mathbb{R}^n$. 
Definition 4.2

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- uniformly stable if there exist a class $\mathcal{K}$ function $\alpha$ and a positive constant $c$, independent of $t_0$, such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \ \forall \ t \geq t_0 \geq 0, \ \forall \ \|x(t_0)\| < c$$

- uniformly asymptotically stable if there exist a class $\mathcal{KL}$ function $\beta$ and a positive constant $c$, independent of $t_0$, such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \ \forall \ t \geq t_0 \geq 0, \ \forall \ \|x(t_0)\| < c$$

- globally uniformly asymptotically stable if the foregoing inequality is satisfied for any initial state $x(t_0)$
- exponentially stable if there exist positive constants $c$, $k$, and $\lambda$ such that
  \[
  \|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \|x(t_0)\| < c
  \]
- globally exponentially stable if the foregoing inequality is satisfied for any initial state $x(t_0)$
Theorem 4.1

Let the origin $x = 0$ be an equilibrium point of $\dot{x} = f(t, x)$ and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Suppose $f(t, x)$ is piecewise continuous in $t$ and locally Lipschitz in $x$ for all $t \geq 0$ and $x \in D$. Let $V(t, x)$ be a continuously differentiable function such that

\[ W_1(x) \leq V(t, x) \leq W_2(x) \]

\[ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0 \]

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on $D$. Then, the origin is uniformly stable.
Theorem 4.2

Suppose the assumptions of the previous theorem are satisfied with

\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x)
\]

for all \( t \geq 0 \) and \( x \in D \), where \( W_3(x) \) is a continuous positive definite function on \( D \). Then, the origin is uniformly asymptotically stable. Moreover, if \( r \) and \( c \) are chosen such that \( B_r = \{\|x\| \leq r\} \subset D \) and \( c < \min_{\|x\|=r} W_1(x) \), then every trajectory starting in \( \{W_2(x) \leq c\} \) satisfies

\[
\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall \ t \geq t_0 \geq 0
\]

for some class \( \mathcal{KL} \) function \( \beta \). Finally, if \( D = \mathbb{R}^n \) and \( W_1(x) \) is radially unbounded, then the origin is globally uniformly asymptotically stable.
Theorem 4.3

Suppose the assumptions of the previous theorem are satisfied with

\[ k_1 \|x\|^a \leq V(t, x) \leq k_2 \|x\|^a \]

\[ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -k_3 \|x\|^a \]

for all \( t \geq 0 \) and \( x \in D \), where \( k_1, k_2, k_3 \), and \( a \) are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable.
Terminology: A function $V(t, x)$ is said to be

- **positive semidefinite** if $V(t, x) \geq 0$
- **positive definite** if $V(t, x) \geq W_1(x)$ for some positive definite function $W_1(x)$
- **radially unbounded** if $V(t, x) \geq W_1(x)$ and $W_1(x)$ is radially unbounded
- **decrescent** if $V(t, x) \leq W_2(x)$
- **negative definite (semidefinite)** if $-V(t, x)$ is positive definite (semidefinite)
Example 4.2

\[ \dot{x} = -[1 + g(t)]x^3, \quad g(t) \geq 0, \quad \forall \ t \geq 0 \]

\[ V(x) = \frac{1}{2}x^2 \]

\[ \dot{V}(t, x) = -[1 + g(t)]x^4 \leq -x^4, \quad \forall \ x \in \mathbb{R}, \quad \forall \ t \geq 0 \]

The origin is globally uniformly asymptotically stable

Example 4.3

\[ \dot{x}_1 = -x_1 - g(t)x_2, \quad \dot{x}_2 = x_1 - x_2 \]

\[ 0 \leq g(t) \leq k \quad \text{and} \quad \dot{g}(t) \leq g(t), \quad \forall \ t \geq 0 \]


\[ V(t, x) = x_1^2 + [1 + g(t)]x_2^2 \]

\[ x_1^2 + x_2^2 \leq V(t, x) \leq x_1^2 + (1 + k)x_2^2, \quad \forall \ x \in \mathbb{R}^2 \]

\[ \dot{V}(t, x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2 \]

\[ 2 + 2g(t) - \dot{g}(t) \geq 2 + 2g(t) - g(t) \geq 2 \]

\[ \dot{V}(t, x) \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = -x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x \]

The origin is globally exponentially stable
Perturbed Systems

Nominal System:
\[ \dot{x} = f(x), \quad f(0) = 0 \]

Perturbed System:
\[ \dot{x} = f(x) + g(t, x), \quad g(t, 0) = 0 \]

Case 1: The origin of the nominal system is exponentially stable

\[ c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2 \]
\[ \frac{\partial V}{\partial x} f(x) \leq -c_3 \|x\|^2 \]
\[ \left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\| \]
Use $V(x)$ as a Lyapunov function candidate for the perturbed system
\[
\dot{V}(t, x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x)
\]
Assume that
\[
\|g(t, x)\| \leq \gamma \|x\|, \quad \gamma \geq 0
\]
\[
\dot{V}(t, x) \leq -c_3 \|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \|g(t, x)\|
\leq -c_3 \|x\|^2 + c_4 \gamma \|x\|^2
\]
\[
\gamma < \frac{c_3}{c_4}
\]

\[
\dot{V}(t, x) \leq -(c_3 - \gamma c_4)\|x\|^2
\]

The origin is an exponentially stable equilibrium point of the perturbed system.
Example 4.4

\[ \dot{x} = Ax + g(t, x); \quad A \text{ is Hurwitz; } \quad \|g(t, x)\| \leq \gamma \|x\| \]

\[ Q = Q^T > 0; \quad PA + A^T P = -Q; \quad V(x) = x^T P x \]

\[ \lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2 \]

\[ \frac{\partial V}{\partial x} Ax = -x^T Q x \leq -\lambda_{\min}(Q)\|x\|^2 \]

\[ \left\| \frac{\partial V}{\partial x} g \right\| = \|2x^T P g\| \leq 2\|P\|\|x\|\|g\| \leq 2\|P\|\gamma\|x\|^2 \]

\[ \dot{V}(t, x) \leq -\lambda_{\min}(Q)\|x\|^2 + 2\lambda_{\max}(P)\gamma\|x\|^2 \]

The origin is globally exponentially stable if \( \gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \)
Example 4.5

\[
\begin{align*}
\dot{x}_1 & = x_2 \\
\dot{x}_2 & = -4x_1 - 2x_2 + \beta x_2^3, \quad \beta \geq 0
\end{align*}
\]

\[
\dot{x} = Ax + g(x)
\]

\[
A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}
\]

The eigenvalues of \( A \) are \(-1 \pm j\sqrt{3}\)

\[
P A + A^T P = -I \quad \Rightarrow \quad P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}
\]
\[ V(x) = x^T P x, \quad \frac{\partial V}{\partial x} Ax = -x^T x \]

\[ c_3 = 1, \quad c_4 = 2 \quad \|P\| = 2\lambda_{\text{max}}(P) = 2 \times 1.513 = 3.026 \]

\[ \|g(x)\| = \beta |x_2|^3 \]

\( g(x) \) satisfies the bound  \( \|g(x)\| \leq \gamma \|x\| \) over compact sets of \( x \). Consider the compact set

\[ \Omega_c = \{ V(x) \leq c \} = \{ x^T P x \leq c \}, \quad c > 0 \]

\[ k_2 = \max_{x^T P x \leq c} |x_2| = \max_{x^T P x \leq c} \left| \begin{bmatrix} 0 & 1 \end{bmatrix} x \right| \]

\[ = \sqrt{c} \left\| \begin{bmatrix} 0 & 1 \end{bmatrix} P^{-1/2} \right\| = 1.8194 \sqrt{c} \]
\[ k_2 = \max_{x^T P x \leq c} |[0 \ 1]x| = 1.8194\sqrt{c} \]

\[ \|g(x)\| \leq \beta \ c \ (1.8194)^2 \|x\|, \quad \forall \ x \in \Omega_c \]

\[ \|g(x)\| \leq \gamma \|x\|, \quad \forall \ x \in \Omega_c, \quad \gamma = \beta \ c \ (1.8194)^2 \]

\[ \gamma < \frac{c_3}{c_4} \iff \beta < \frac{1}{3.026 \times (1.8194)^2 c} \approx \frac{0.1}{c} \]

\[ \beta < 0.1/c \iff \dot{V}(x) \leq -(1 - 10\beta c)\|x\|^2 \]

Hence, the origin is exponentially stable and \( \Omega_c \) is an estimate of the region of attraction.
Alternative Bound on $\beta$:

$$\dot{V}(x) = -\|x\|^2 + 2x^TPg(x) \leq -\|x\|^2 + \frac{1}{8}\beta x_2^3 ([2 \ 5]x)$$

$$\leq -\|x\|^2 + \sqrt{29} \beta x_2^2 \|x\|^2$$

Over $\Omega_c$, $x_2^2 \leq (1.8194)^2 c$

$$\dot{V}(x) \leq - \left( 1 - \frac{\sqrt{29}}{8} \beta (1.8194)^2 c \right) \|x\|^2$$

$$= - \left( 1 - \frac{\beta c}{0.448} \right) \|x\|^2$$

If $\beta < 0.448/c$, the origin will be exponentially stable and $\Omega_c$ will be an estimate of the region of attraction.
Remark

The inequality $\beta < 0.448/c$ shows a tradeoff between the estimate of the region of attraction and the estimate of the upper bound on $\beta$. 
Case 2: The origin of the nominal system is asymptotically stable

\[ \dot{V}(t, x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x) \leq -W_3(x) + \left\| \frac{\partial V}{\partial x} g(t, x) \right\| \]

Under what condition will the following inequality hold?

\[ \left\| \frac{\partial V}{\partial x} g(t, x) \right\| < W_3(x) \]

Special Case: Quadratic-Type Lyapunov function

\[ \frac{\partial V}{\partial x} f(x) \leq -c_3 \phi^2(x), \quad \left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \phi(x) \]
\[ \dot{V}(t, x) \leq -c_3 \phi^2(x) + c_4 \phi(x) \|g(t, x)\| \]

If \[ \|g(t, x)\| \leq \gamma \phi(x), \quad \text{with} \quad \gamma < \frac{c_3}{c_4} \]

\[ \dot{V}(t, x) \leq -(c_3 - c_4 \gamma)\phi^2(x) \]
Example 4.6

\[ \dot{x} = -x^3 + g(t, x) \]

\[ V(x) = x^4 \] is a quadratic-type Lyapunov function for \( \dot{x} = -x^3 \)

\[ \frac{\partial V}{\partial x}(-x^3) = -4x^6, \quad \left| \frac{\partial V}{\partial x} \right| = 4|x|^3 \]

\[ \phi(x) = |x|^3, \quad c_3 = 4, \quad c_4 = 4 \]

Suppose \( |g(t, x)| \leq \gamma |x|^3, \ \forall \ x, \ \text{with} \ \gamma < 1 \)

\[ \dot{V}(t, x) \leq -4(1 - \gamma)\phi^2(x) \]

Hence, the origin is a globally uniformly asymptotically stable
Remark

A nominal system with asymptotically, but not exponentially, stable origin is not robust to smooth perturbations with arbitrarily small linear growth bounds

Example 4.7

\[
\dot{x} = -x^3 + \gamma x
\]

The origin is unstable for any \( \gamma > 0 \)