Passivity-Based Control

In Section 9.6 we saw that if the system

$$\dot{x} = f(x, u), \quad y = h(x)$$

is passive (with a positive definite storage function) and zero-state observable, it can be stabilized by

$$u = -\phi(y), \quad \phi(0) = 0, \quad y^T \phi(y) > 0, \quad \forall \ y \neq 0$$

Suppose the system

$$\dot{x} = f(x, u), \quad \dot{y} = \frac{\partial h}{\partial x} f(x, u) \overset{\text{def}}{=} \tilde{h}(x, u)$$

is passive (with a positive definite storage function $V(x)$) and zero state observable.
Nonlinear Control Lecture # 13 Output Feedback Stabilization
\[
\frac{s}{\tau s + 1}
\]

\[
\tau \dot{w} = -w + y, \quad z = (-w + y)/\tau
\]

MIMO systems

\[
\tau_i \dot{w}_i = -w_i + y_i, \quad z_i = (-w_i + y_i)/\tau_i, \quad \text{for } 1 \leq i \leq m
\]

Note that

\[
\tau_i \dot{z}_i = -z_i + \dot{y}_i
\]
Lemma 12.1

Consider the system

\[
\dot{x} = f(x, u), \quad y = h(x)
\]

and the output feedback controller

\[
u_i = -\phi_i(z_i), \quad \tau_i \dot{w}_i = -w_i + y_i, \quad z_i = (-w_i + y_i)/\tau_i
\]

\[
\tau_i > 0, \quad \phi_i(0) = 0, \quad z_i \phi_i(z_i) > 0 \quad \forall \quad z_i \neq 0
\]

Suppose the auxiliary system

\[
\dot{x} = f(x, u), \quad \dot{y} = \tilde{h}(x, u)
\]
passive with a positive definite storage function \( V(x) \)

\[
u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u), \quad \forall (x, u)
\]

zero-state observable

with \( u = 0 \), \( \dot{y}(t) \equiv 0 \) \( \Rightarrow \) \( x(t) \equiv 0 \)

Then the origin of the closed-loop system is asymptotically stable. It is globally asymptotically stable if \( V(x) \) is radially unbounded and \( \int_{z_i}^{\infty} \phi_i(\sigma) \, d\sigma \to \infty \) as \( |z_i| \to \infty \)
Proof

\[ W(x, z) = V(x) + \sum_{i=1}^{m} \tau_i \int_{0}^{z_i} \phi_i(\sigma) \, d\sigma \]

\[ \dot{W} = \dot{V} + \sum_{i=1}^{m} \tau_i \phi_i(z_i) \dot{z}_i \leq u^T \dot{y} - \sum_{i=1}^{m} z_i \phi_i(z_i) - u^T \dot{y} \]

\[ \dot{W} \leq - \sum_{i=1}^{m} z_i \phi_i(z_i) \]

\[ \dot{W} \equiv 0 \implies z(t) \equiv 0 \implies u(t) \equiv 0 \quad \text{and} \quad \dot{y}(t) \equiv 0 \]

Apply the invariance principle
Example 12.2 \((m\text{-link Robot Manipulator})\)

\[
M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + D \dot{q} + g(q) = u
\]

\[
M = M^T > 0, \quad (\dot{M} - 2C)^T = -(\dot{M} - 2C), \quad D = D^T \geq 0
\]

Stabilize the system at \(q = q_r, \quad e = q - q_r, \quad \dot{e} = \dot{q}\)

\[
M(q) \ddot{e} + C(q, \dot{q}) \dot{e} + D \dot{e} + g(q) = u
\]

\[
u = g(q) - K_p e + v, \quad [K_p = K_p > 0]
\]

\[
M(q) \ddot{e} + C(q, \dot{q}) \dot{e} + D \dot{e} + K_p e = v, \quad y = e
\]

\[
V = \frac{1}{2} \dot{e}^T M(q) \dot{e} + \frac{1}{2} e^T K_p e
\]
\[ V = \frac{1}{2} \dot{e}^T M(q) \dot{e} + \frac{1}{2} e^T K_p e \]

\[ \dot{V} = \frac{1}{2} \dot{e}^T (\dot{M} - 2C) \dot{e} - \dot{e}^T D \dot{e} - \dot{e}^T K_p e + \dot{e}^T v + e^T K_p \dot{e} \leq \dot{e}^T v \]

Is it zero-state observable? Set \( v = 0 \)

\[ \dot{e}(t) \equiv 0 \Rightarrow \ddot{e}(t) \equiv 0 \Rightarrow K_p e(t) \equiv 0 \Rightarrow e(t) \equiv 0 \]

\[ \tau_i \dot{w}_i = -w_i + e_i, \quad z_i = (-a_i w_i + e_i)/\tau_i, \quad \text{for } 1 \leq i \leq m \]

\[ u = g(q) - K_p (q - q_r) - K_d z \]

\( K_d \) is positive diagonal matrix. Compare with state feedback

\[ u = g(q) - K_p (q - q_r) - K_d \dot{q} \]
Observer-Based Control

\[
\dot{x} = f(x, u), \quad y = h(x)
\]

**State Feedback Controller:** Design a locally Lipschitz \( u = \gamma(x) \) to stabilize the origin of

\[
\dot{x} = f(x, \gamma(x))
\]

**Observer:**

\[
\dot{x} = f(\hat{x}, u) + H[y - h(\hat{x})]
\]

\[
\tilde{x} = x - \hat{x}
\]

\[
\dot{x} = f(x, u) - f(\hat{x}, u) - H[h(x) - h(\hat{x})] \overset{\text{def}}{=} g(x, \tilde{x})
\]

\[
g(x, 0) = 0
\]
Design $H$ such that $\dot{x} = g(x, \tilde{x})$ has an exponentially stable equilibrium point at $\tilde{x} = 0$ and there is Lyapunov function $V_1(\tilde{x})$ such that

$$c_1 \|\tilde{x}\|^2 \leq V_1 \leq c_2 \|\tilde{x}\|^2, \quad \frac{\partial V_1}{\partial \tilde{x}} g \leq -c_3 \|\tilde{x}\|^2, \quad \left\| \frac{\partial V_1}{\partial \tilde{x}} \right\| \leq c_4 \|\tilde{x}\|$$

$$u = \gamma(\hat{x})$$

Closed-loop system:

$$\dot{x} = f(x, \gamma(x - \tilde{x})), \quad \hat{x} = g(x, \tilde{x}) \quad (\star)$$
Theorem 12.1

- If the origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable, so is the origin of $(\star)$
- If the origin of $\dot{x} = f(x, \gamma(x))$ is exponentially stable, so is the origin of $(\star)$
- If the assumptions hold globally and the system $\dot{x} = f(x, \gamma(x - \tilde{x}))$, with input $\tilde{x}$, is ISS, then the origin of $(\star)$ is globally asymptotically stable
High-Gain Observers

Example 12.3

\[
\begin{align*}
\dot{x}_1 &= x_2, \quad \dot{x}_2 = \phi(x, u), \quad y = x_1 \\
\end{align*}
\]

State feedback control: \( u = \gamma(x) \) stabilizes the origin of

\[
\begin{align*}
\dot{x}_1 &= x_2, \quad \dot{x}_2 = \phi(x, \gamma(x)) \\
\end{align*}
\]

High-gain observer

\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 + \left(\frac{\alpha_1}{\varepsilon}\right)(y - \hat{x}_1), \quad \dot{\hat{x}}_2 = \phi_0(\hat{x}, u) + \left(\frac{\alpha_2}{\varepsilon^2}\right)(y - \hat{x}_1) \\
\end{align*}
\]

\( \phi_0 \) is a nominal model of \( \phi \), \( \alpha_i > 0, \ 0 < \varepsilon \ll 1 \)

\[
\begin{align*}
|\tilde{x}_1| &\leq \max \left\{ b e^{-at/\varepsilon}, \varepsilon^2 cM \right\}, \quad |\tilde{x}_2| \leq \left\{ \frac{b}{\varepsilon} e^{-at/\varepsilon}, \varepsilon cM \right\}
\end{align*}
\]
The bound on $\tilde{x}_2$ demonstrates the peaking phenomenon, which might destabilize the closed-loop system

**Example:**

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3^2 + u, \quad y = x_1
\]

**State feedback control:**

\[
u = -x_2^3 - x_1 - x_2
\]

**Output feedback control:**

\[
u = -\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2
\]

\[
\dot{\hat{x}}_1 = \hat{x}_2 + \frac{2}{\varepsilon}(y - \hat{x}_1), \quad \dot{\hat{x}}_2 = \frac{1}{\varepsilon^2}(y - \hat{x}_1)
\]
Nonlinear Control Lecture # 13 Output Feedback Stabilization
$\epsilon = 0.004$
Closed-loop system under state feedback:

\[ \dot{x} = Ax, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \]

\[ PA + A^T P = -I \quad \Rightarrow \quad P = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix} \]

Suppose \( x(0) \) belongs to the positively invariant set \( \Omega = \left\{ V(x) \leq 0.3 \right\} \)

\[ |u| \leq |x_2|^3 + |x_1 + x_2| \leq 0.816, \quad \forall \ x \in \Omega \]

Saturate \( u \) at ±1
$u = \text{sat}(-\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2)$
Region of attraction under state feedback:
Region of attraction under output feedback:

\[ \varepsilon = 0.08 \text{ (dashed) and } \varepsilon = 0.01 \text{ (dash-dot)} \]
Analysis of the closed-loop system:

\[ \dot{x}_1 = x_2 \]
\[ \varepsilon \dot{\eta}_1 = -\alpha_1 \eta_1 + \eta_2 \]
\[ \varepsilon \dot{\eta}_2 = -\alpha_2 \eta_1 + \varepsilon \delta(x, \tilde{x}) \]
General case

\[ \dot{w} = \psi(w, x, u) \]
\[ \dot{x}_i = x_{i+1} + \psi_i(x_1, \ldots, x_i, u), \quad 1 \leq i \leq \rho - 1 \]
\[ \dot{x}_\rho = \phi(w, x, u) \]
\[ y = x_1 \]
\[ z = q(w, x) \]

\[ \phi(0, 0, 0) = 0, \quad \psi(0, 0, 0) = 0, \quad q(0, 0) = 0 \]

The normal form and models of mechanical and electromechanical systems take this form with

\[ \psi_1 = \cdots = \psi_\rho = 0 \]

Why the extra measurement \( z \)?
Stabilizing state feedback controller:

\[ \dot{\vartheta} = \Gamma(\vartheta, x, z), \quad u = \gamma(\vartheta, x, z) \]

\( \gamma \) and \( \Gamma \) are globally bounded functions of \( x \)

Closed-loop system

\[ \dot{X} = f(X), \quad X = \text{col}(w, x, \vartheta) \]

Output feedback controller

\[ \dot{\hat{\vartheta}} = \Gamma(\vartheta, \hat{x}, z), \quad u = \gamma(\vartheta, \hat{x}, z) \]
Observer

\[
\dot{x}_i = \dot{x}_{i+1} + \psi_i(\hat{x}_1, \ldots, \hat{x}_i, u) + \frac{\alpha_i}{\varepsilon_i}(y - \hat{x}_1), \quad 1 \leq i \leq \rho - 1
\]

\[
\dot{x}_\rho = \phi_0(z, \hat{x}, u) + \frac{\alpha_\rho}{\varepsilon_\rho}(y - \hat{x}_1)
\]

\(\varepsilon > 0\) and \(\alpha_1\) to \(\alpha_\rho\) are chosen such that the roots of

\[
s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1} s + \alpha_\rho = 0
\]

have negative real parts
Theorem 12.2
Suppose the origin of $\dot{X} = f(X)$ is asymptotically stable and $R$ is its region of attraction. Let $S$ be any compact set in the interior of $R$ and $Q$ be any compact subset of $R^\rho$. Then, given any $\mu > 0$ there exist $\varepsilon^* > 0$ and $T^* > 0$, dependent on $\mu$, such that for every $0 < \varepsilon \leq \varepsilon^*$, the solutions $(X(t), \hat{x}(t))$ of the closed-loop system, starting in $S \times Q$, are bounded for all $t \geq 0$ and satisfy

$$\|X(t)\| \leq \mu \quad \text{and} \quad \|\hat{x}(t)\| \leq \mu, \quad \forall \ t \geq T^*$$

$$\|X(t) - X_r(t)\| \leq \mu, \quad \forall \ t \geq 0$$

where $X_r$ is the solution of $\dot{X} = f(X)$, starting at $X(0)$.
If the origin of $\dot{X} = f(X')$ is exponentially stable, then the origin of the closed-loop system is exponentially stable and $S \times Q$ is a subset of its region of attraction.
Robust Stabilization of Minimum Phase Systems

Relative Degree One

\[ \dot{\eta} = f_0(\eta, y), \quad \dot{y} = a(\eta, y) + b(\eta, y)u + \delta(t, \eta, y, u) \]

\[ f_0(0, 0) = 0, \quad a(0, 0) = 0, \quad b(\eta, y) \geq b_0 > 0 \]

The origin of \( \dot{\eta} = f_0(\eta, 0) \) is asymptotically stable

\[ \alpha_1(\|\eta\|) \leq V(\eta) \leq \alpha_2(\|\eta\|) \]

\[ \frac{\partial V}{\partial \eta} f_0(\eta, y) \leq -\alpha_3(\|\eta\|), \quad \forall \|\eta\| \geq \alpha_4(|y|) \]

**Sliding Mode Control:** Sliding surface \( y = 0 \)

\[ u = \psi(y) + v \]
\[ \left| \frac{a(\eta, y) + b(\eta, y)\psi(y) + \delta(t, \eta, y, \psi(y) + v)}{b(\eta, y)} \right| \leq \rho(y) + \kappa_0 |v| \]

\[ 0 \leq \kappa_0 < 1 \]

\[ \beta(y) \geq \frac{\rho(y)}{1 - \kappa_0} + \beta_0 \]

\[ v = -\beta(y) \text{ sat} \left( \frac{y}{\mu} \right) \]

\[ u = \psi(y) - \beta(y) \text{ sat} \left( \frac{y}{\mu} \right) \]

All the assumptions hold in a domain \( D \)
Relative Degree Higher Than One

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, \quad \text{for } 1 \leq i \leq \rho - 1 \\
\dot{\xi}_\rho &= a(\eta, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) \\
y &= \xi_1 \\
\end{align*}
\]

\[f_0(0, 0) = 0, \quad a(0, 0) = 0, \quad b(\eta, \xi) \geq b_0 > 0\]

The origin of \( \dot{\eta} = f_0(\eta, 0) \) is asymptotically stable

**Partial State Feedback:** Assume \( \xi \) is available for feedback

\[s = k_1\xi_1 + k_2\xi_2 + \cdots + k_{\rho-1}\xi_{\rho-1} + \xi_\rho\]
With \( s \) as the output, the system has relative degree one and the normal form is given by

\[
\dot{z} = \bar{f}_0(z, s), \quad \dot{s} = \bar{a}(z, s) + \bar{b}(z, s)u + \bar{\delta}(t, z, s, u)
\]

\[
z = \text{col} \left( \eta, \xi_1, \ldots, \xi_{\rho-2}, \xi_{\rho-1} \right)
\]

Zero Dynamics \((s = 0)\):

\[
\dot{z} = \bar{f}_0(z, 0)
\]
\[ \dot{z} = \bar{f}_0(z, 0) \]

\[ \iff \quad \dot{\eta} = f_0(\eta, \xi) \quad \text{subject to} \quad x_\rho = -\sum_{i=1}^{\rho-1} k_i \xi_i \]

\[ \zeta = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{\rho-1} \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ -k_1 & -k_2 & \cdots & -k_{\rho-2} & -k_{\rho-1} \end{bmatrix} \]

When \( \rho = n \), the zero dynamics are \( \dot{\zeta} = F\zeta \)
$k_1$ to $k_{\rho-1}$ are chosen such that the polynomial

$$\lambda^{\rho-1} + k_{\rho-1}\lambda^{\rho-2} + \cdots + k_2\lambda + k_1$$

is Hurwitz

$$\alpha_1(\|z\|) \leq V(z) \leq \alpha_2(\|z\|)$$

$$\partial V \bar{f}_0(z, s) \leq -\alpha_3(\|z\|), \quad \forall \|z\| \geq \alpha_4(|s|)$$

We have converted the relative degree $\rho$ system into a relative degree one system that satisfies the earlier assumptions

$$u = \psi(\xi) + v$$
\[ \frac{\bar{a}(z, s) + \bar{b}(z, s)\psi(\xi) + \delta(t, z, s, \psi(\xi) + v)}{\bar{b}(z, s)} \leq \rho(\xi) + \kappa_0|v| \]

Left hand side equals
\[ \left| \sum_{i=1}^{\rho-1} k_i \xi_{i+1} + a(\eta, \xi) + b(\eta, \xi)\psi(\xi) + \delta(t, \eta, \xi, \psi(\xi) + v) \right| \]

\[ \beta(\xi) \geq \frac{\rho(\xi)}{1 - \kappa_0} + \beta_0, \quad \beta_0 > 0 \]

\[ u = \psi(\xi) - \beta(\xi) \text{ sat} \left( \frac{s}{\mu} \right) \]

Saturate \( \beta \) and \( \psi \)