Nonlinear Control
Lecture # 9
Time Varying and Perturbed Systems
Perturbed Systems

Nominal System:
\[ \dot{x} = f(x), \quad f(0) = 0 \]

Perturbed System:
\[ \dot{x} = f(x) + g(t, x), \quad g(t, 0) = 0 \]

Case 1: The origin of the nominal system is exponentially stable
\[ c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2 \]
\[ \frac{\partial V}{\partial x} f(x) \leq -c_3 \|x\|^2 \]
\[ \left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\| \]
Use $V(x)$ as a Lyapunov function candidate for the perturbed system

$$\dot{V}(t, x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x)$$

Assume that

$$\|g(t, x)\| \leq \gamma \|x\|, \quad \gamma \geq 0$$

$$\dot{V}(t, x) \leq -c_3 \|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \|g(t, x)\|$$

$$\leq -c_3 \|x\|^2 + c_4 \gamma \|x\|^2$$
The origin is an exponentially stable equilibrium point of the perturbed system

\[
\gamma < \frac{c_3}{c_4}
\]

\[
\dot{V}(t, x) \leq -(c_3 - \gamma c_4) \|x\|^2
\]
Example 4.4

\[
\dot{x} = Ax + g(t, x); \quad A \text{ is Hurwitz}; \quad \|g(t, x)\| \leq \gamma \|x\|
\]

\[
Q = Q^T > 0; \quad PA + A^TP = -Q; \quad V(x) = x^TPx
\]

\[
\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2
\]

\[
\frac{\partial V}{\partial x}Ax = -x^TQx \leq -\lambda_{\min}(Q)\|x\|^2
\]

\[
\left\|\frac{\partial V}{\partial x}g\right\| = \|2x^TPg\| \leq 2\|P\|\|x\|\|g\| \leq 2\|P\|\gamma\|x\|^2
\]

\[
\dot{V}(t, x) \leq -\lambda_{\min}(Q)\|x\|^2 + 2\lambda_{\max}(P)\gamma\|x\|^2
\]

The origin is globally exponentially stable if \( \gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} \)
Example 4.5

\[\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3, \quad \beta \geq 0
\end{align*}\]

\[\dot{x} = Ax + g(x)\]

\[A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}\]

The eigenvalues of \(A\) are \(-1 \pm j\sqrt{3}\)

\[PA + A^TP = -I \quad \Rightarrow \quad P = \begin{bmatrix} 3 & 1 \\ 2 & 8 \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}\]
\[ V(x) = x^T Px, \quad \frac{\partial V}{\partial x} Ax = -x^T x \]

\[ c_3 = 1, \quad c_4 = 2 \quad \|P\| = 2\lambda_{\text{max}}(P) = 2 \times 1.513 = 3.026 \]

\[ \|g(x)\| = \beta |x_2|^3 \]

\( g(x) \) satisfies the bound \( \|g(x)\| \leq \gamma \|x\| \) over compact sets of \( x \). Consider the compact set

\[ \Omega_c = \{ V(x) \leq c \} = \{ x^T Px \leq c \}, \quad c > 0 \]

\[ k_2 = \max_{x^T Px \leq c} |x_2| = \max_{x^T Px \leq c} \left| \begin{bmatrix} 0 & 1 \end{bmatrix} x \right| \]

\[ = \sqrt{c} \left\| \begin{bmatrix} 0 & 1 \end{bmatrix} P^{-1/2} \right\| = 1.8194\sqrt{c} \]
\[ k_2 = \max_{x^T P x \leq c} |[0 \ 1] x| = 1.8194 \sqrt{c} \]

\[ \|g(x)\| \leq \beta c (1.8194)^2 \|x\|, \quad \forall \ x \in \Omega_c \]

\[ \|g(x)\| \leq \gamma \|x\|, \quad \forall \ x \in \Omega_c, \quad \gamma = \beta c (1.8194)^2 \]

\[ \gamma < \frac{c_3}{c_4} \iff \beta < \frac{1}{3.026 \times (1.8194)^2 c} \approx \frac{0.1}{c} \]

\[ \beta < \frac{0.1}{c} \Rightarrow \dot{V}(x) \leq -(1 - 10\beta c) \|x\|^2 \]

Hence, the origin is exponentially stable and \( \Omega_c \) is an estimate of the region of attraction.
Alternative Bound on $\beta$:

$$
\dot{V}(x) = -\|x\|^2 + 2x^T Pg(x) \leq -\|x\|^2 + \frac{1}{8} \beta x_2^3 \begin{bmatrix} 2 & 5 \end{bmatrix} x \\
\leq -\|x\|^2 + \frac{\sqrt{29}}{8} \beta x_2^2 \|x\|^2
$$

Over $\Omega_c$, $x_2^2 \leq (1.8194)^2 c$

$$
\dot{V}(x) \leq - \left( 1 - \frac{\sqrt{29}}{8} \beta (1.8194)^2 c \right) \|x\|^2 \\
= - \left( 1 - \frac{\beta c}{0.448} \right) \|x\|^2
$$

If $\beta < 0.448/c$, the origin will be exponentially stable and $\Omega_c$ will be an estimate of the region of attraction.
Remark

The inequality $\beta < 0.448/c$ shows a tradeoff between the estimate of the region of attraction and the estimate of the upper bound on $\beta$. 
Case 2: The origin of the nominal system is asymptotically stable

\[ \dot{V}(t, x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x) \leq -W_3(x) + \left\| \frac{\partial V}{\partial x} g(t, x) \right\| \]

Under what condition will the following inequality hold?

\[ \left\| \frac{\partial V}{\partial x} g(t, x) \right\| < W_3(x) \]

Special Case: Quadratic-Type Lyapunov function

\[ \frac{\partial V}{\partial x} f(x) \leq -c_3 \phi^2(x), \quad \left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \phi(x) \]
\[ \dot{V}(t, x) \leq -c_3 \phi^2(x) + c_4 \phi(x) \|g(t, x)\| \]

If \[ \|g(t, x)\| \leq \gamma \phi(x), \] with \[ \gamma < \frac{c_3}{c_4} \]

\[ \dot{V}(t, x) \leq -(c_3 - c_4 \gamma) \phi^2(x) \]
Example 4.6

\[
\dot{x} = -x^3 + g(t, x)
\]

\(V(x) = x^4\) is a quadratic-type Lyapunov function for \(\dot{x} = -x^3\)

\[
\frac{\partial V}{\partial x}(-x^3) = -4x^6, \quad \left| \frac{\partial V}{\partial x} \right| = 4|x|^3
\]

\[
\phi(x) = |x|^3, \quad c_3 = 4, \quad c_4 = 4
\]

Suppose \(|g(t, x)| \leq \gamma |x|^3, \quad \forall \ x, \quad \text{with } \gamma < 1\)

\[
\dot{V}(t, x) \leq -4(1 - \gamma)\phi^2(x)
\]

Hence, the origin is a globally uniformly asymptotically stable
Remark

A nominal system with asymptotically, but not exponentially, stable origin is not robust to smooth perturbations with arbitrarily small linear growth bounds.

Example 4.7

\[ \dot{x} = -x^3 + \gamma x \]

The origin is unstable for any \( \gamma > 0 \).