Nonlinear Control
Lecture # 35
Output Feedback Stabilization
Robust Stabilization of Minimum Phase Systems

Relative Degree One

\[ \dot{\eta} = f_0(\eta, y), \quad \dot{y} = a(\eta, y) + b(\eta, y)u + \delta(t, \eta, y, u) \]

\[ f_0(0, 0) = 0, \quad a(0, 0) = 0, \quad b(\eta, y) \geq b_0 > 0 \]

The origin of \( \dot{\eta} = f_0(\eta, 0) \) is asymptotically stable

\[ \alpha_1(\|\eta\|) \leq V(\eta) \leq \alpha_2(\|\eta\|) \]

\[ \frac{\partial V}{\partial \eta} f_0(\eta, y) \leq -\alpha_3(\|\eta\|), \quad \forall \|\eta\| \geq \alpha_4(|y|) \]

**Sliding Mode Control:** Sliding surface \( y = 0 \)

\[ u = \psi(y) + v \]
\[
\left| \frac{a(\eta, y) + b(\eta, y) \psi(y) + \delta(t, \eta, y, \psi(y) + v)}{b(\eta, y)} \right| \leq \varrho(y) + \kappa_0 |v|
\]

0 \leq \kappa_0 < 1

\[
\beta(y) \geq \frac{\varrho(y)}{1 - \kappa_0} + \beta_0
\]

\[
v = -\beta(y) \text{ sat} \left( \frac{y}{\mu} \right)
\]

\[
u = \psi(y) - \beta(y) \text{ sat} \left( \frac{y}{\mu} \right)
\]

All the assumptions hold in a domain \( D \)
Theorem 12.3

Define the class $\mathcal{K}$ function $\alpha$ by $\alpha(r) = \alpha_2(\alpha_4(r))$ and suppose $\mu$, $c > \mu$, and $c_0 \geq \alpha(c)$ are chosen such that the set

$$\Omega = \{V(\eta) \leq c_0\} \times \{|y| \leq c\}, \quad \text{with } c_0 \geq \alpha(c)$$

is compact and contained in $D$. Then, $\Omega$ is positively invariant and for any initial state in $\Omega$, the state is bounded for all $t \geq 0$ and reaches the positively invariant set

$$\Omega_\mu = \{V(\eta) \leq \alpha(\mu)\} \times \{|y| \leq \mu\}$$

in finite time. Moreover, if the assumptions hold globally and $V(\eta)$ is radially unbounded, the foregoing conclusion holds for any initial state.
Theorem 12.4

Suppose \( \varrho(0) = 0 \) and the origin of \( \dot{\eta} = f_0(\eta, 0) \) is exponentially stable. Then, there exists \( \mu^* > 0 \) such that for all \( 0 < \mu < \mu^* \), the origin of the closed-loop system is exponentially stable and \( \Omega \) is a subset of its region of attraction. Moreover, if the assumptions hold globally and \( V(\eta) \) is radially unbounded, the origin will be globally uniformly asymptotically stable.
Relative Degree Higher Than One

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, \quad \text{for } 1 \leq i \leq \rho - 1 \\
\dot{\xi}_\rho &= a(\eta, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) \\
y &= \xi_1
\end{align*}
\]

\[
f_0(0, 0) = 0, \quad a(0, 0) = 0, \quad b(\eta, \xi) \geq b_0 > 0
\]

The origin of \( \dot{\eta} = f_0(\eta, 0) \) is asymptotically stable

**Partial State Feedback:** Assume \( \xi \) is available for feedback

\[
s = k_1 \xi_1 + k_2 \xi_2 + \cdots + k_{\rho-1} \xi_{\rho-1} + \xi_\rho
\]
With $s$ as the output, the system has relative degree one and the normal form is given by

$$
\dot{z} = \vec{f}_0(z, s), \quad \dot{s} = \vec{a}(z, s) + \vec{b}(z, s)u + \vec{\delta}(t, z, s, u)
$$

$$
z = \text{col} \left( \eta, \ \dot{\xi}_1, \ \ldots, \ \dot{\xi}_{\rho-2}, \ \dot{\xi}_{\rho-1} \right)
$$

Zero Dynamics ($s = 0$):

$$
\dot{z} = \vec{f}_0(z, 0)
$$
\[
\dot{\zeta} = \bar{f}_0(z, 0)
\]

\[
\Leftrightarrow \quad \dot{\eta} = f_0(\eta, \xi) \quad \begin{array}{c}
\xi_\rho = -\sum_{i=1}^{\rho-1} k_i \xi_i
\end{array}, \quad \dot{\zeta} = F\zeta
\]

\[
\zeta = \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_{\rho-1}
\end{bmatrix}, \quad F = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & & & & \vdots \\
0 & \cdots & 0 & 1 \\
-k_1 & -k_2 & \cdots & -k_{\rho-2} & -k_{\rho-1}
\end{bmatrix}
\]

When \( \rho = n \), the zero dynamics are \( \dot{\zeta} = F\zeta \)
$k_1$ to $k_{\rho-1}$ are chosen such that the polynomial

$$\lambda^{\rho-1} + k_{\rho-1}\lambda^{\rho-2} + \cdots + k_2\lambda + k_1$$

is Hurwitz

$$\alpha_1(\|z\|) \leq V(z) \leq \alpha_2(\|z\|)$$

$$\frac{\partial V}{\partial \eta} \tilde{f}_0(z, s) \leq -\alpha_3(\|z\|), \quad \forall \|z\| \geq \alpha_4(|s|)$$

We have converted the relative degree $\rho$ system into a relative degree one system that satisfies the earlier assumptions

$$u = \psi(\xi) + v$$
\[
\left| \frac{\bar{a}(z, s) + \bar{b}(z, s)\psi(\xi) + \bar{\delta}(t, z, s, \psi(\xi) + v)}{\bar{b}(z, s)} \right| \leq \rho(\xi) + \kappa_0 |v|
\]

Left hand side equals

\[
\left| \sum_{i=1}^{\rho-1} k_i \xi_{i+1} + a(\eta, \xi) + b(\eta, \xi)\psi(\xi) + \delta(t, \eta, \xi, \psi(\xi) + v) \right|\]

\[\beta(\xi) \geq \frac{\rho(\xi)}{1 - \kappa_0} + \beta_0, \quad \beta_0 > 0\]

\[u = \psi(\xi) - \beta(\xi) \text{ sat } \left( \frac{s}{\mu} \right) = \gamma(\xi)\]
Saturation

Scheme 1:

\[ M_i = \max_{\Omega} \{|\xi_i|\}, \quad 1 \leq i \leq \rho \]

\[ \psi_s(\xi) = \psi(\xi) \quad \xi_i = M_i \text{ sat} \left( \frac{\xi_i}{M_i} \right) \]

\[ \beta_s(\xi) = \beta(\xi) \quad \xi_i = M_i \text{ sat} \left( \frac{\xi_i}{M_i} \right) \]

Scheme 2:

\[ M_\psi = \max_{\Omega} \{|\psi(\xi)|\}, \quad M_\beta = \max_{\Omega} \{|\beta(\xi)|\} \]

\[ \psi_s(\xi) = M_\psi \text{ sat} (\psi(\xi)/M_\psi), \quad \beta_s(\xi) = M_\beta \text{ sat} (\beta(\xi)/M_\beta) \]
\[ u = \psi_s(\xi) - \beta_s(\xi) \operatorname{sat} \left( \frac{s}{\mu} \right) \]

\( \beta_s \) and \( \psi_s \) are globally bounded functions of \( \xi \)

Scheme 3:

\[ M_u = \max_{\Omega} \left\{ |\psi(\xi) - \beta(x) \operatorname{sat}(s/\mu)| \right\} \]

\[ u = M_u \operatorname{sat} \left( \frac{\psi(\xi) - \beta(\xi) \operatorname{sat}(s/\mu)}{M_u} \right) \]
Output Feedback Controller

\[
\dot{\hat{\xi}}_i = \hat{\xi}_{i+1} + \frac{\alpha_i}{\varepsilon_i} (y - \hat{\xi}_1), \quad 1 \leq i \leq \rho - 1
\]

\[
\dot{\hat{\xi}}_\rho = a_0(\hat{\xi}) + b_0(\hat{\xi})u + \frac{\alpha_\rho}{\varepsilon_\rho} (y - \hat{\xi}_1)
\]

\[s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1} s + \alpha_\rho \text{ is Hurwitz}\]

\[u = \gamma_s(\hat{\xi}), \quad \gamma_s(\hat{\xi}) \text{ is given by}\]

\[
\psi_s(\hat{\xi}) - \beta_s(\hat{\xi}) \text{ sat}(\frac{\hat{s}}{\mu}) \quad \text{or} \quad M_u \text{ sat} \left( \frac{\psi(\hat{\xi}) - \beta(\hat{\xi}) \text{ sat}(\hat{s}/\mu)}{M_u} \right)
\]

\[
\hat{s} = \sum_{i=1}^{\rho-1} k_i \hat{\xi}_i + \hat{\xi}_\rho
\]

Nonlinear Control Lecture # 35 Output Feedback Stabilization
Theorem 12.5

Let \( \Omega_0 \) be a compact set in the interior of \( \Omega \), \( X \) a compact subset of \( R^\rho \), \( (\eta(0), \xi(0)) \in \Omega_0 \), and \( \hat{\xi}(0) \in X \). Then, there exists \( \varepsilon^* \), dependent on \( \mu \), such that for all \( \varepsilon \in (0, \varepsilon^*) \) the states \( (\eta(t), \xi(t), \hat{\xi}(t)) \) of the closed-loop system are bounded for all \( t \geq 0 \) and there is a finite time \( T \), dependent on \( \mu \), such that \( (\eta(t), \xi(t)) \in \Omega_\mu \) for all \( t \geq T \). Moreover, given any \( \lambda > 0 \), there exists \( \varepsilon^{**} > 0 \), dependent on \( \mu \) and \( \lambda \), such that for all \( \varepsilon \in (0, \varepsilon^{**}) \),

\[
\| \eta(t) - \eta_r(t) \| \leq \lambda \quad \text{and} \quad \| \xi(t) - \xi_r(t) \| \leq \lambda, \quad \forall \ t \in [0, T]
\]

where \( (\eta_r(t), \xi_r(t)) \) is the state of the closed-loop system under state feedback with initial conditions \( \eta_r(0) = \eta(0) \) and \( \xi_r(0) = \xi(0) \).
Theorem 12.6

Suppose all the assumptions of Theorem 12.5 are satisfied. Then, there exists $\mu^* > 0$ and for each $\mu \in (0, \mu^*)$ there exists $\varepsilon^* > 0$, dependent on $\mu$, such that for all $\varepsilon \in (0, \varepsilon^*)$, the origin of the closed-loop system under output feedback is exponentially stable and $\Omega_0 \times X$ is a subset of its region of attraction.
Example 12.4 (Pendulum Equation)

\[ \ddot{\theta} + \sin \theta + b \dot{\theta} = cu \]

\[ 0 \leq b \leq 0.2, \quad 0.5 \leq c \leq 2 \]

From Example 10.1: \( u = -2(|x_1| + |x_2| + 1) \text{ sat } (s/\mu) \)

stabilizes the pendulum at \((\theta = \pi, \dot{\theta} = 0)\)

Suppose now that we only measure \(\theta\). In preparation for using a high-gain observer, we will saturate the state feedback control outside a compact set

\[ \Omega = \{|x_1| \leq c/\theta_1\} \times \{|s| \leq c\}, \quad c > 0, \quad 0 < \theta_1 < 1 \]

Take \(c = 2\pi\) and \(\theta = 0.8\)
\[ |x_1| \leq 2.5\pi, \quad |x_2| \leq 4.5\pi, \quad \forall x \in \Omega \]

**Output Feedback Controller:**

\[
u = -2 \left( 2.5\pi \ \text{sat} \left( \frac{|\hat{x}_1|}{2.5\pi} \right) + 4.5\pi \ \text{sat} \left( \frac{|\hat{x}_2|}{4.5\pi} \right) + 1 \right) \ \text{sat} \left( \frac{s}{\mu} \right)\]

\[
\hat{s} = \hat{x}_1 + \hat{x}_2
\]

\[
\hat{x}_1 = \hat{x}_2 + \frac{2}{\varepsilon}(x_1 - \hat{x}_1), \quad \hat{x}_2 = \phi_0(\hat{x}, u) + \frac{1}{\varepsilon^2}(x_1 - \hat{x}_1)
\]

**Simulation:**

\[
b = 0.01, \quad c = 0.5, \quad x_1(0) = -\pi, \quad x_2(0) = \hat{x}_i(0) = 0
\]

\[
\phi_0(\hat{x}) = \begin{cases} 
0 & \text{Figures (a) & (b)} \\
-\sin(\hat{x}_1 + \pi) - 0.1\hat{x}_2 + 1.25u & \text{Figures (c) & (d)}
\end{cases}
\]
\[ \theta \]

(a) SF
- OF \( \varepsilon = 0.05 \)
- OF \( \varepsilon = 0.01 \)

(b) \( \omega \)

(c) \( \theta \)

(d) \( \omega \)

Time

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