Nonlinear Control
Lecture # 28
Robust State Feedback Stabilization
Sliding Mode Control

\[ \dot{x} = f(x) + B(x)[G(x)u + \delta(t, x, u)] \]

\[ x \in \mathbb{R}^n, \; u \in \mathbb{R}^m, \; f \text{ and } B \text{ are known, while } G \text{ and } \delta \text{ could be uncertain, } f(0) = 0, \; G(x) \text{ is a positive definite symmetric matrix with} \]

\[ \lambda_{min}(G(x)) \geq \lambda_0 > 0 \]

Regular Form:

\[
\begin{bmatrix}
\eta \\
\xi
\end{bmatrix} = T(x), \quad \frac{\partial T}{\partial x} B(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}
\]

\[ \dot{\eta} = f_a(\eta, \xi), \quad \dot{\xi} = f_b(\eta, \xi) + G(x)u + \delta(t, x, u) \]
\[ \dot{\eta} = f_a(\eta, \xi), \quad \dot{\xi} = f_b(\eta, \xi) + G(x)u + \delta(t, x, u) \]

**Sliding Manifold:**

\[ s = \xi - \phi(\eta) = 0, \quad \phi(0) = 0 \]

\[ s(t) \equiv 0 \Rightarrow \dot{\eta} = f_a(\eta, \phi(\eta)) \]

Design \( \phi \) s.t. the origin of \( \dot{\eta} = f_a(\eta, \phi(\eta)) \) is asymptotically stable.
\[ \dot{s} = f_b(\eta, \xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + G(x)u + \delta(t, x, u) \]

\[ u = \psi(\eta, \xi) + v \]

Typical choices of \( \psi \):

\[ \psi = 0, \quad \psi = -\hat{G}^{-1}[f_b - (\partial \phi / \partial \eta) f_a] \]

\[ \dot{s} = G(x)v + \Delta(t, x, v) \]

\[ \left\| \frac{\Delta(t, x, v)}{\lambda_{\text{min}}(G(x))} \right\| \leq \varrho(x) + \kappa_0 \|v\|, \quad \forall (t, x, v) \in [0, \infty) \times D \times \mathbb{R}^m \]

\[ \varrho(x) \geq 0, \quad 0 \leq \kappa_0 < 1 \quad (\text{Known}) \]
\[ V = \frac{1}{2} s^T s \Rightarrow \dot{V} = s^T \dot{s} = s^T G(x)v + s^T \Delta(t, x, v) \]

\[ v = -\beta(x) \frac{s}{\|s\|}, \quad \beta(x) \geq \frac{\varrho(x)}{1 - \kappa_0} + \beta_0, \quad \beta_0 > 0 \]

\[ \dot{V} = -\beta(x) s^T G(x)s/\|s\| + s^T \Delta(t, x, v) \leq \lambda_{\text{min}}(G(x))[-\beta(x) + \varrho(x) + \kappa_0 \beta(x)] \|s\| \]

\[ = \lambda_{\text{min}}(G(x))[-(1 - \kappa_0)\beta(x) + \varrho(x)] \|s\| \leq -\lambda_{\text{min}}(G(x)) \beta_0 (1 - \kappa_0) \|s\| \leq -\lambda_0 \beta_0 (1 - \kappa_0) \|s\| = -\lambda_0 \beta_0 (1 - \kappa_0) \sqrt{2V} \]

Trajectories reach the manifold \( s = 0 \) in finite time and cannot leave it.
Continuous Implementation

$$\text{Sat}(y) = \begin{cases} 
  y, & \text{if } \|y\| \leq 1 \\
  y/\|y\|, & \text{if } \|y\| > 1 
\end{cases}$$

$$v = -\beta(x) \text{ Sat} \left( \frac{s}{\mu} \right)$$

$$\|s\| \geq \mu \Rightarrow \text{Sat}(s/\mu) = s/\|s\| \Rightarrow s^T \dot{s} \leq -\lambda_0 \beta_0 (1 - \kappa_0) \|s\|$$

Trajectories reach the boundary layer \(\{\|s\| \leq \mu\}\) in finite time and remains inside thereafter

Study the behavior of \(\eta\): \(\dot{\eta} = f_a(\eta, \phi(\eta) + s)\)
\[ \alpha_1(\| \eta \|) \leq V_0(\eta) \leq \alpha_2(\| \eta \|) \]

\[ \frac{\partial V_0}{\partial \eta} f_a(\eta, \phi(\eta) + s) \leq -\alpha_3(\| \eta \|), \quad \forall \| \eta \| \geq \alpha_4(\| s \|) \]

\[ \| s \| \leq c \Rightarrow \dot{V}_0 \leq -\alpha_3(\| \eta \|), \quad \text{for} \quad \| \eta \| \geq \alpha_4(c) \]

\[ \alpha(r) = \alpha_2(\alpha_4(r)) \]

\[ V_0(\eta) \geq \alpha(c) \Leftrightarrow V_0(\eta) \geq \alpha_2(\alpha_4(c)) \Rightarrow \alpha_2(\| \eta \|) \geq \alpha_2(\alpha_4(c)) \]

\[ \Rightarrow \| \eta \| \geq \alpha_4(c) \]

\[ \Rightarrow \dot{V}_0 \leq -\alpha_3(\| \eta \|) \leq -\alpha_3(\alpha_4(c)) \]

\[ \Omega = \{ V_0(\eta) \leq c_0 \} \times \{ \| s \| \leq c \}, \quad c_0 \geq \alpha(c), \quad \Omega \subset T(D) \]
\[ V_0(\eta) \geq \alpha(\mu) \implies \dot{V}_0 \leq -\alpha_3(\alpha_4(\mu)) \]

\[ \implies \Omega_\mu = \{V_0(\eta) \leq \alpha(\mu)\} \times \{\|s\| \leq \mu\} \text{ is positively invariant} \]

In summary, all trajectories starting in \( \Omega \) remain in \( \Omega \) and reach \( \Omega_\mu \) in finite time and remain inside thereafter.
Theorem 10.1

Suppose all the assumptions hold over $\Omega$. Then, for all $(\eta(0), \xi(0)) \in \Omega$, the trajectory $(\eta(t), \xi(t))$ is bounded for all $t \geq 0$ and reaches the positively invariant set $\Omega_\mu$ in finite time. If the assumptions hold globally and $V(\eta)$ is radially unbounded, the foregoing conclusion holds for any initial state.
Example 10.2 (Magnetic levitation - friction neglected)

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = 1 + \frac{m_o}{m} u, \quad x_1 \geq 0, \quad -2 \leq u \leq 0
\]

We want to stabilize the system at \( x_1 = 1 \). Nominal steady-state control is \( u_{ss} = -1 \)

Shift the equilibrium point to the origin: \( x_1 \rightarrow x_1 - 1, \ u \rightarrow u + 1 \)

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{m - m_o}{m} + \frac{m_o}{m} u
\]

\( x_1 \geq -1, \quad |u| \leq 1 \)

Assume \( \left| \frac{(m - m_o)}{m_o} \right| \leq \frac{1}{3} \)
\[ s = x_1 + x_2 \quad \Rightarrow \quad \dot{x}_1 = -x_1 + s \]

\[ V_0 = \frac{1}{2} x_1^2 \]

\[ \dot{V}_0 = -x_1^2 + x_1 s \leq -(1-\theta)x_1^2, \quad \forall |x_1| \geq |s|/\theta, \quad 0 < \theta < 1 \]

\[ \alpha_1(r) = \alpha_2(r) = \frac{1}{2} r^2, \quad \alpha_3(r) = (1-\theta)r^2, \quad \alpha_4(r) = r/\theta \]

\[ \alpha(r) = \alpha_2(\alpha_4(r)) = \frac{1}{2}(r/\theta)^2 \]

With \( c_0 = \alpha(c), \quad \Omega = \{|x_1| \leq c/\theta\} \times \{|s| \leq c\} \)

\[ \Omega_\mu = \{|x_1| \leq \mu/\theta\} \times \{|s| \leq \mu\} \]
\[ \Omega = \{ |x_1| \leq c/\theta \} \times \{ |s| \leq c \} \]

Take \( c \leq \theta \) to meet the constraint \( x_1 \geq -1 \)

\[ \dot{s} = x_2 + \frac{m - m_o}{m} + \frac{m_o}{m} u \]

\[
\left| \frac{x_2 + (m - m_o)/m}{m_o/m} \right| = \left| \frac{m}{m_o} x_2 + \frac{m - m_o}{m_o} \right| \leq \frac{1}{3} (4|x_2| + 1)
\]

In \( \Omega \), \( |x_2| \leq |x_1| + |s| \leq c(1 + 1/\theta) \)

with \( \frac{1}{\theta} = 1.1 \), \( \left| \frac{x_2 + (m - m_o)/m}{m_o/m} \right| \leq \frac{8.4c + 1}{3} \)
To meet the constraint $|u| \leq 1$ limit $c$ to

$$\frac{8.4c + 1}{3} < 1 \Leftrightarrow c < 0.238 \text{ and take } u = -\text{sat}\left(\frac{s}{\mu}\right)$$

With $c = 0.23$, Theorem 10.1 ensures that all trajectories starting in $\Omega$ stay in $\Omega$ and enter $\Omega_{\mu}$ in finite time

Inside $\Omega_{\mu}$, $|x_1| \leq \mu/\theta = 1.1\mu$

$\mu$ can be chosen small enough to meet any specified ultimate bound on $x_1$

For $|x_1| \leq 0.01$, take $\mu = 0.01/1.1 \approx 0.009$
With further analysis inside $\Omega_\mu$ we can derive a less conservative estimate of the ultimate bound of $|x_1|$. In $\Omega_\mu$, the closed-loop system is represented by

$$
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{m - m_o}{m} - \frac{m_o(x_1 + x_2)}{m\mu}
\end{align*}
$$

which has a unique equilibrium point at

$$
\left( x_1 = \frac{\mu(m - m_o)}{m_o}, \ x_2 = 0 \right)
$$

and its matrix is Hurwitz

$$
\lim_{t \to \infty} x_1(t) = \frac{\mu(m - m_o)}{m_o}, \quad \lim_{t \to \infty} x_2(t) = 0
$$
\[
\left| \frac{(m - m_o)}{m_o} \right| \leq \frac{1}{3} \Rightarrow |x_1| \leq 0.34\mu
\]

For \(|x_1| \leq 0.01\), take \(\mu = 0.029\)

We can also obtain a less conservative estimate of the region of attraction

\[
V_1 = \frac{1}{2}(x_1^2 + s^2)
\]

\[
\dot{V}_1 \leq -x_1^2 + s^2 - \frac{m_o}{m} \left[ 1 - \left| \frac{m - m_o}{m_o} \right| \right] |s| \leq -x_1^2 + s^2 - \frac{1}{2}|s|
\]

for \(|s| \geq \mu\)
\[ \dot{V}_1 \leq -x_1^2 + s^2 + \left| \frac{m - m_0}{m_o} \right| |s| - \frac{m_o s^2}{m \mu} \leq -x_1^2 + s^2 + \frac{1}{2}|s| - \frac{3s^2}{4\mu} \]

for \(|s| \leq \mu\)

With \(\mu = 0.029\), it can be verified that \(\dot{V}_1\) is less than a negative number in the set \(\{0.0012 \leq V_1 \leq 0.12\}\). Therefore, all trajectories starting in \(\Omega_1 = \{V_1 \leq 0.12\}\) enter \(\Omega_2 = \{V_1 \leq 0.0012\}\) in finite time. Since \(\Omega_2 \subset \Omega\), our earlier analysis holds and the ultimate bound of \(|x_1|\) is 0.01. The new estimate of the region of attraction, \(\Omega_1\), is larger than \(\Omega\).
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