Nonlinear Control

Lecture # 18

Stability of Feedback Systems
Absolute Stability

Definition 7.1

The system is absolutely stable if the origin is globally uniformly asymptotically stable for any nonlinearity in a given sector. It is absolutely stable with finite domain if the origin is uniformly asymptotically stable.
Circle Criterion

Suppose \( G(s) = C(sI - A)^{-1}B + D \) is SPR, \( \psi \in [0, \infty] \)

\[
\dot{x} = Ax + Bu \\
y = Cx + Du \\
u = -\psi(t, y)
\]

By the KYP Lemma, \( \exists P = P^T > 0, L, W, \varepsilon > 0 \)

\[
PA + A^TP = -L^TL - \varepsilon P \\
PB = C^T - L^TW \\
W^TW = D + D^T
\]

\[
V(x) = \frac{1}{2}x^TPx
\]
\[
\dot{V} = \frac{1}{2} x^T P \dot{x} + \frac{1}{2} \dot{x}^T P x
\]
\[
= \frac{1}{2} x^T (PA + A^T P)x + x^T PBu
\]
\[
= -\frac{1}{2} x^T L^T L x - \frac{1}{2} \varepsilon x^T P x + x^T (C^T - L^T W) u
\]
\[
= -\frac{1}{2} x^T L^T L x - \frac{1}{2} \varepsilon x^T P x + (Cx + Du)^T u
\]
\[
- u^T Du - x^T L^T W u
\]

\[
u^T Du = \frac{1}{2} u^T (D + D^T) u = \frac{1}{2} u^T W^T W u
\]
\[
\dot{V} = -\frac{1}{2} \varepsilon x^T P x - \frac{1}{2} (Lx + W u)^T (Lx + W u) - y^T \psi(t, y)
\]
\[
y^T \psi(t, y) \geq 0 \quad \Rightarrow \quad \dot{V} \leq -\frac{1}{2} \varepsilon x^T P x
\]

The origin is globally exponentially stable
What if $\psi \in [K_1, \infty]$?

$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $G(s)[I + K_1 G(s)]^{-1}$ is SPR
What if \( \psi \in [K_1, K_2] \)?

\[
\begin{align*}
\psi & \in [0, \infty]; \text{ hence the origin is globally exponentially stable if} \\
I + KG(s)[I + K_1G(s)]^{-1} \text{ is SPR}
\end{align*}
\]
Theorem 7.8 (Circle Criterion)

The system is absolutely stable if

- \( \psi \in [K_1, \infty] \) and \( G(s)[I + K_1G(s)]^{-1} \) is SPR, or
- \( \psi \in [K_1, K_2] \) and \( [I + K_2G(s)][I + K_1G(s)]^{-1} \) is SPR

If the sector condition is satisfied only on a set \( Y \subset \mathbb{R}^m \), then the foregoing conditions ensure absolute stability with finite domain.

\[
I + KG(s)[I + K_1G(s)]^{-1} = [I + K_2G(s)][I + K_1G(s)]^{-1}
\]
Example 7.11

$G(s)$ is Hurwitz, $G(\infty) = 0$

$$\|\psi(t, y)\| \leq \gamma_2 \|y\|, \ \forall \ t, \ y$$

$$\psi \in [K_1, K_2], \quad K_1 = -\gamma_2 I, \quad K_2 = \gamma_2 I$$

By the circle criterion, the system is absolutely stable if

$$Z(s) = [I + \gamma_2 G(s)][I - \gamma_2 G(s)]^{-1}$$

is SPR

$$Z(\infty) + Z^T(\infty) = 2I$$

By Lemma 5.1,

$$Z(s) \text{ SPR } \iff Z(s) \text{ Hurwitz } \& Z(j\omega) + Z^T(-j\omega) > 0 \ \forall \ \omega$$

$$Z(s) \text{ Hurwitz } \iff [I - \gamma_2 G(s)]^{-1} \text{ Hurwitz}$$
From the multivariable Nyquist criterion, \([I - \gamma_2 G(s)]^{-1}\) is Hurwitz if the plot of \(\det[I - \gamma_2 G(j\omega)]\) does not go through nor encircle the origin. This will be the case if

\[
\sigma_{\min}[I - \gamma_2 G(j\omega)] > 0
\]

\[
\sigma_{\min}[I - \gamma_2 G(j\omega)] \geq 1 - \gamma_1 \gamma_2, \quad \gamma_1 = \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|
\]

\[
\gamma_1 \gamma_2 < 1 \Rightarrow Z(s) \text{ Hurwitz}
\]

\[
Z(j\omega) + Z^T(-j\omega) = 2H^T(-j\omega) \left[I - \gamma_2^2 G^T(-j\omega)G(j\omega)\right] H(j\omega)
\]

\[
H(j\omega) = [I - \gamma_2 G(j\omega)]^{-1}
\]

\[
Z(j\omega) + Z^T(-j\omega) > 0 \Leftrightarrow \left[I - \gamma_2^2 G^T(-j\omega)G(j\omega)\right] > 0
\]

\[
\gamma_1 \gamma_2 < 1 \Rightarrow Z(s) \text{ SPR}
\]
Scalar Case: $\psi \in [\alpha, \beta], \beta > \alpha$

The system is absolutely stable if

$$\frac{1 + \beta G(s)}{1 + \alpha G(s)}$$

is Hurwitz and

$$\text{Re} \left[ \frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)} \right] > 0, \ \forall \omega \in [0, \infty]$$
Case 1: $\alpha > 0$

By the Nyquist criterion

$$\frac{1 + \beta G(s)}{1 + \alpha G(s)} = \frac{1}{1 + \alpha G(s)} + \frac{\beta G(s)}{1 + \alpha G(s)}$$

is Hurwitz if the Nyquist plot of $G(j\omega)$ does not intersect the point $-(1/\alpha) + j0$ and encircles it $p$ times in the counterclockwise direction, where $p$ is the number of poles of $G(s)$ in the open right-half complex plane

$$\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)} > 0 \iff \frac{1}{\beta} + \frac{G(j\omega)}{1 + \alpha G(j\omega)} > 0$$
The system is absolutely stable if the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha, \beta)$ and encircles it $m$ times in the counterclockwise direction.
Case 2: $\alpha = 0$

$$1 + \beta G(s)$$

$$\text{Re}[1 + \beta G(j\omega)] > 0, \quad \forall \omega \in [0, \infty]$$

$$\text{Re}[G(j\omega)] > -\frac{1}{\beta}, \quad \forall \omega \in [0, \infty]$$

The system is absolutely stable if $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies to the right of the vertical line defined by $\text{Re}[s] = -1/\beta$
Case 3: \( \alpha < 0 < \beta \)

\[
\text{Re} \left[ \frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)} \right] > 0 \iff \text{Re} \left[ \frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)} \right] < 0
\]

The Nyquist plot of \( G(j\omega) \) must lie inside the disk \( D(\alpha, \beta) \). The Nyquist plot cannot encircle the point \(-1/\alpha + j0\).

From the Nyquist criterion, \( G(s) \) must be Hurwitz

The system is absolutely stable if \( G(s) \) is Hurwitz and the Nyquist plot of \( G(j\omega) \) lies in the interior of the disk \( D(\alpha, \beta) \).
Theorem 7.9

Consider an SISO $G(s)$ and $\psi \in [\alpha, \beta]$. Then, the system is absolutely stable if one of the following conditions is satisfied.

1. $0 < \alpha < \beta$, the Nyquist plot of $G(s)$ does not enter the disk $D(\alpha, \beta)$ and encircles it $p$ times in the counterclockwise direction, where $p$ is the number of poles of $G(s)$ with positive real parts.

2. $0 = \alpha < \beta$, $G(s)$ is Hurwitz and the Nyquist plot of $G(s)$ lies to the right of the vertical line $\text{Re}[s] = -1/\beta$.

3. $\alpha < 0 < \beta$, $G(s)$ is Hurwitz and the Nyquist plot of $G(s)$ lies in the interior of the disk $D(\alpha, \beta)$.

If the sector condition is satisfied only on an interval $[a, b]$, then the foregoing conditions ensure absolute stability with finite domain.
Example 7.12

\[ G(s) = \frac{24}{(s + 1)(s + 2)(s + 3)} \]
Apply Case 3 with center \((0, 0)\) and radius \(= 4\)

Sector is \((-0.25, 0.25)\)

Apply Case 3 with center \((1.5, 0)\) and radius \(= 2.9\)

Sector is \([-0.227, 0.714]\)

Apply Case 2

The Nyquist plot is to the right of \(\text{Re}[s] = -0.857\)

Sector is \([0, 1.166]\)

\([0, 1.166]\) includes the saturation nonlinearity
Example 7.13

\[ G(s) = \frac{24}{(s - 1)(s + 2)(s + 3)} \]

\[ G \] is not Hurwitz

Apply Case 1

Center = \((-3.2, 0)\), Radius = 0.1688 \Rightarrow [0.2969, 0.3298]
Example 7.14

\[ G(s) = \frac{s + 2}{(s + 1)(s - 1)}, \quad \psi(y) = \text{sat}(y) \in [0, 1] \]

We cannot conclude absolute stability because case 1 requires \( \alpha > 0 \)

\[-a \leq y \leq a \Rightarrow \psi \in [\alpha, 1], \quad \alpha = \frac{1}{a}\]
The Nyquist plot must encircle the disk $D(1/a, 1)$ once in the counterclockwise direction, which is satisfied for $a = 1.818$

The system is absolutely stable with finite domain
Estimate the region of attraction:

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 + u, \quad y = 2x_1 + x_2 \]

Loop transformation:

\[ u = -\alpha y + \tilde{u}, \quad \tilde{y} = (\beta - \alpha) y + \tilde{u}, \quad \alpha = \frac{1}{a} = 0.55, \quad \beta = 1 \]

\[ \dot{x} = Ax + B\tilde{u}, \quad \tilde{y} = Cx + D\tilde{u} \]

where

\[ A = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.55 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.9 & 0.45 \end{bmatrix}, \quad D = 1 \]
The KYP equations have two solutions

\[ P_1 = \begin{bmatrix} 0.4946 & 0.4834 \\ 0.4834 & 1.0774 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.7595 & 0.4920 \\ 0.4920 & 1.9426 \end{bmatrix} \]

\[ V_1(x) = x^T P_1 x, \quad V_2(x) = x^T P_2 x \]

\[ \min\{\|y\|=1.818\} \quad V_1(x) = 0.3445, \quad \min\{\|y\|=1.818\} \quad V_2(x) = 0.6212 \]

\[ \{V_1(x) \leq 0.34\}, \quad \{V_2(x) \leq 0.62\} \]
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$V_1(x) = 0.34$

$V_2(x) = 0.62$

$y = 1.818$

$y = -1.818$