Nonlinear Control Lecture # 16 Stability of Feedback Systems

\[
\dot{x}_i = f_i(x_i, e_i), \quad y_i = h_i(x_i, e_i)
\]

\[
y_i = h_i(t, e_i)
\]
Passivity Theorems

Theorem 7.1
The feedback connection of two passive systems is passive

Proof
Let \( V_1(x_1) \) and \( V_2(x_2) \) be the storage functions for \( H_1 \) and \( H_2 \) (\( V_i = 0 \) if \( H_i \) is memoryless)

\[
e_i^T y_i \geq \dot{V}_i, \quad V(x) = V_1(x_1) + V_2(x_2)
\]

\[
e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2
\]

\[
u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]

\[
u^T y = u_1^T y_1 + u_2^T y_2 \geq \dot{V}_1 + \dot{V}_2 = \dot{V}
\]
Asymptotic Stability

Theorem 7.2

Consider the feedback connection of two dynamical systems. When $u = 0$, the origin of the closed-loop system is asymptotically stable if one of the following conditions is satisfied:

- both feedback components are strictly passive;
- both feedback components are output strictly passive and zero-state observable;
- one component is strictly passive and the other one is output strictly passive and zero-state observable.

If the storage function for each component is radially unbounded, the origin is globally asymptotically stable.
Proof

$H_1$ is SP; $H_2$ is OSP & ZSO

\[ e_1^T y_1 \geq \dot{V}_1 + \psi_1(x_1), \quad \psi_1(x_1) > 0, \; \forall \; x_1 \neq 0 \]

\[ e_2^T y_2 \geq \dot{V}_2 + y_2^T \rho_2(y_2), \quad y_2^T \rho(y_2) > 0, \; \forall \; y_2 \neq 0 \]

\[ e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2 \]

\[ V(x) = V_1(x_1) + V_2(x_2) \]

\[ \dot{V} \leq u^T y - \psi_1(x_1) - y_2^T \rho_2(y_2) \]

\[ u = 0 \Rightarrow \dot{V} \leq -\psi_1(x_1) - y_2^T \rho_2(y_2) \]
\[ \dot{V} \leq -\psi_1(x_1) - y_2^T \varrho_2(y_2) \]

\[ \dot{V} = 0 \implies x_1 = 0 \text{ and } y_2 = 0 \]

\[ y_2(t) \equiv 0 \implies e_1(t) \equiv 0 \quad (\& \quad x_1(t) \equiv 0) \implies y_1(t) \equiv 0 \]

\[ y_1(t) \equiv 0 \implies e_2(t) \equiv 0 \]

By zero-state observability of \( H_2 \):

\[ y_2(t) \equiv 0 \implies x_2(t) \equiv 0 \]

Apply the invariance principle
Example 7.1

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -ax_1^3 - kx_2 + e_1 \\
y_1 &= x_2
\end{align*}
\]

\[
\begin{align*}
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -bx_3 - x_4^3 + e_2 \\
y_2 &= x_4
\end{align*}
\]

\[a, \, b, \, k > 0\]

\[
V_1 = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2
\]

\[
\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - kx_2^2 + x_2e_1 = -ky_1^2 + y_1e_1
\]

With \(e_1 = 0, \, y_1(t) \equiv 0 \iff x_2(t) \equiv 0 \implies x_1(t) \equiv 0\)

\(H_1\) is output strictly passive and zero-state observable
\[ V_2 = \frac{1}{2} bx_3^2 + \frac{1}{2} x_4^2 \]

\[ \dot{V}_2 = bx_3 x_4 - bx_3 x_4 - x_4^4 + x_4 e_2 = -y_2^4 + y_2 e_2 \]

With \( e_2 = 0 \), \( y_2(t) \equiv 0 \iff x_4(t) \equiv 0 \Rightarrow x_3(t) \equiv 0 \)

\( H_2 \) is output strictly passive and zero-state observable

\( V_1 \) and \( V_2 \) are radially unbounded

The origin is globally asymptotically stable
Example 7.2

Reconsider the previous example, but change the output of \( H_1 \) to

\[
y_1 = x_2 + e_1
\]

\[
\dot{V}_1 = -kx_2^2 + x_2e_1 = -k(y_1 - e_1)^2 - e_1^2 + y_1e_1
\]

\( H_1 \) is passive, but we cannot conclude strict passivity or output strict passivity. We cannot apply Theorem 7.2.

\[
V = V_1 + V_2 = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2 + \frac{1}{2}bx_3^2 + \frac{1}{2}x_4^2
\]

\[
\dot{V} = -kx_2^2 + x_2e_1 - x_4^4 + x_4e_2
\]

\[
= -kx_2^2 - x_2x_4 - x_4^4 + x_4(x_2 - x_4)
\]

\[
= -kx_2^2 - x_4^4 - x_4^2 \leq 0
\]
\[ \dot{V} = -kx_2^2 - x_4^4 - x_4^2 \leq 0 \]

\[ \dot{V} = 0 \Rightarrow x_2 = x_4 = 0 \]

\[ x_2(t) \equiv 0 \Rightarrow ax_1^3(t) - x_4(t) \equiv 0 \Rightarrow x_1(t) \equiv 0 \]

\[ x_4(t) \equiv 0 \Rightarrow -bx_3(t) + x_2(t) - x_4(t) \equiv 0 \Rightarrow x_3(t) \equiv 0 \]

By the invariance principle and the fact that \( V \) is radially unbounded, we conclude that the origin is globally asymptotically stable.
Example 7.3

Reconsider

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -h_1(x_1) - h_2(x_2), \quad h_i \in (0, \infty) \]

from Example 3.8.
\[ H_1 : \quad \dot{x}_2 = -h_2(x_2) + e_1, \quad y_1 = x_2 \]

\[ V_1 = \frac{1}{2} x_2^2, \quad \dot{V}_1 = -x_2 h_2(x_2) + x_2 e_1 = -y_1 h_1(y_1) + y_1 e_1 \]

Output Strictly Passive (Strictly Passive)

\[ H_2 : \quad \dot{x}_1 = e_2, \quad y_2 = h_1(x_1) \]

\[ V_2 = \int_0^{x_1} h_1(\sigma) \, d\sigma, \quad \dot{V}_2 = h_1(x_1) e_2 = y_2 e_2 \quad \text{Lossless} \]

We cannot apply Theorem 7.2, but use

\[ V = V_1 + V_2 = \int_0^{x_1} h_1(\sigma) \, d\sigma + \frac{1}{2} x_2^2 \]

as a Lyapunov function candidate (Examples 3.8 and 3.9)
Theorem 7.3

Consider the feedback connection of a strictly passive dynamical system with a passive time-varying memoryless function. When \( u = 0 \), the origin of the closed-loop system is uniformly asymptotically stable. If the storage function for the dynamical system is radially unbounded, the origin will be globally uniformly asymptotically stable.

Proof

Let \( V_1(x_1) \) be (positive definite) storage function of \( H_1 \).

\[
\dot{V}_1 = \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) \leq e_1^T y_1 - \psi_1(x_1) = -e_2^T y_2 - \psi_1(x_1)
\]

\[
e_2^T y_2 \geq 0 \quad \Rightarrow \quad \dot{V}_1 \leq -\psi_1(x_1)
\]
Example 7.4

Consider the feedback connection of a strictly positive real transfer function and a passive time-varying memoryless function.

From Lemma 5.4, we know that the dynamical system is strictly passive with a positive definite storage function

$$V(x) = \frac{1}{2}x^TPx$$

From Theorem 7.3, the origin of the closed-loop system is globally uniformly asymptotically stable.
Theorem 7.4

Consider the feedback connection of a time-invariant dynamical system $H_1$ with a time-invariant memoryless function $H_2$. Suppose $H_1$ is zero-state observable, $V_1(x_1)$ is positive definite

$$e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1), \quad e_2^T y_2 \geq e_2^T \varphi_2(e_2)$$

Then, the origin of the closed-loop system (when $u = 0$) is asymptotically stable if

$$v^T [\rho_1(v) + \varphi_2(v)] > 0, \quad \forall \ v \neq 0$$

Furthermore, if $V_1$ is radially unbounded, the origin will be globally asymptotically stable
Example 7.5

\[ \dot{x} = f(x) + G(x)e_1 \]
\[ y_1 = h(x) \]
\[ y_2 = \sigma(e_2) \]
\[ H_1 \]
\[ H_2 \]
\[ \sigma(0) = 0, \quad e_2^T \sigma(e_2) > 0, \quad \forall \ e_2 \neq 0 \]

Suppose \( H_1 \) is zero-state observable and there is a radially unbounded positive definite function \( V_1(x) \) such that

\[ \frac{\partial V_1}{\partial x} f(x) \leq 0, \quad \frac{\partial V_1}{\partial x} G(x) = h^T(x), \quad \forall \ x \in \mathbb{R}^n \]

\[ \dot{V}_1 = \frac{\partial V_1}{\partial x} f(x) + \frac{\partial V_1}{\partial x} G(x)e_1 \leq y_1^T e_1 \]
Apply Theorem 7.4:

\[ \dot{V}_1 \leq e_1^T y_1 \]

\[ e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1) \text{ is satisfied with } \rho_1 = 0 \]

\[ e_2^T y_2 = e_2^T \sigma(e_2) \]

\[ e_2^T y_2 \geq e_2^T \varphi_2(e_2) \text{ is satisfied with } \varphi_2 = \sigma \]

\[ v^T [\rho_1(v) + \varphi_2(v)] = v^T \sigma(v) > 0, \quad \forall \ v \neq 0 \]

The origin is globally asymptotically stable