September 22, 2000

Useful Formulae for Electromagnetics

By: Leo Kempel (kempel@egr.msu.edu)

Coordinate Transformation

Cylindrical-to-Cartesian and Spherical-to-Cartesian

$$x = \rho\cos(\phi) = r\sin(\theta)\cos(\phi)$$
$$y = \rho\sin(\phi) = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

Cartesian-to-Cylindrical and Spherical-to-Cylindrical

$$\rho = \sqrt{x^2 + y^2} = r\sin(\theta)$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = r\cos(\theta)$$

Cartesian-to-Spherical and Cylindrical-to-Spherical

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = \tan^{-1}\left(\frac{\rho}{z}\right)$$

Component Transformation

To-Cartesian: $\mathbf{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

$$A_x = A_\rho\cos(\phi) - A_\phi\sin(\phi) = A_r\sin(\theta)\cos(\phi) + A_\theta\cos(\theta)\cos(\phi) - A_\phi\sin(\phi)$$
$$A_y = A_\rho\sin(\phi) + A_\phi\cos(\phi) = A_r\sin(\theta)\sin(\phi) + A_\theta\cos(\theta)\sin(\phi) + A_\phi\cos(\phi)$$
$$A_z = A_r\cos(\theta) - A_\theta\sin(\theta)$$

To-Cylindrical: $\mathbf{A} = A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}$

$$A_\rho = A_x\cos(\phi) + A_y\sin(\phi) = A_r\sin(\theta) + A_\theta\cos(\theta)$$
$$A_\phi = -A_x\sin(\phi) + A_y\cos(\phi)$$
$$A_z = A_r\cos(\theta) - A_\theta\sin(\theta)$$

To-Spherical: $\mathbf{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

$$A_r = A_x\sin(\theta)\cos(\phi) + A_y\sin(\theta)\sin(\phi) + A_z\cos(\theta) = A_\rho\sin(\theta) + A_z\cos(\theta)$$
$$A_\theta = A_x\cos(\theta)\cos(\phi) + A_y\cos(\theta)\sin(\phi) - A_z\sin(\theta) = A_\rho\cos(\theta) - A_z\sin(\theta)$$
$$A_\phi = -A_x\sin(\phi) + A_y\cos(\phi)$$

Unit Vector Transformation

To-Cartesian:

$$\hat{x} = \hat{\rho}\cos(\phi) - \hat{\phi}\sin(\phi) = \hat{r}\sin(\theta)\cos(\phi) + \hat{\theta}\cos(\theta)\cos(\phi) - \hat{\phi}\sin(\phi)$$
$$\hat{y} = \hat{\rho}\sin(\phi) + \hat{\phi}\cos(\phi) = \hat{r}\sin(\theta)\sin(\phi) + \hat{\theta}\sin(\theta)\cos(\phi) + \hat{\phi}\sin(\phi)$$
$$\hat{z} \qquad = \hat{r}\cos(\theta) - \hat{\theta}\sin(\theta)$$

To-Cylindrical:

$$\hat{\rho} = \hat{x}\cos(\phi) + \hat{y}\sin(\phi) = \hat{r}\sin(\theta) + \hat{\theta}\cos(\theta)$$
$$\hat{\phi} = -\hat{x}\sin(\phi) + \hat{y}\cos(\phi)$$
$$\hat{z} \qquad = \hat{r}\cos(\theta) - \hat{\theta}\sin(\theta)$$

To-Spherical:

$$\hat{r} = \hat{x}\sin(\theta)\cos(\phi) + \hat{y}\sin(\theta)\sin(\phi) + \hat{z}\cos(\theta) = \hat{\rho}\sin(\theta) + \hat{z}\cos(\theta)$$
$$\hat{\theta} = \hat{x}\cos(\theta)\cos(\phi) + \hat{y}\cos(\theta)\sin(\phi) - \hat{z}\sin(\theta) = \hat{\rho}\cos(\theta) - \hat{z}\sin(\theta)$$
$$\hat{\phi} = -\hat{x}\sin(\phi) + \hat{y}\cos(\phi)$$

Differentiation

Cartesian: $w(x,y,z)$ and $\mathbf{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

$$\nabla w = \hat{x}\frac{\partial w}{\partial x} + \hat{y}\frac{\partial w}{\partial y} + \hat{z}\frac{\partial w}{\partial z}$$
$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$\nabla \times \mathbf{A} = \hat{x}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \hat{y}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \hat{z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$
$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

Cylindrical: $w(\rho,\phi,z)$ and $\mathbf{A} = A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}$

$$\nabla w = \hat{\rho}\frac{\partial w}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial w}{\partial \phi} + \hat{z}\frac{\partial w}{\partial z}$$
$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$
$$\nabla \times \mathbf{A} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{z}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)$$
$$\nabla^2 w = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial w}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{\partial^2 w}{\partial z^2}$$

Spherical: $w(r,\theta,\phi)$ and $\mathbf{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

$$\nabla w = \hat{r}\frac{\partial w}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial w}{\partial \theta} + \hat{\phi}\frac{1}{r\sin(\theta)}\frac{\partial w}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}\left[\sin(\theta)A_\theta\right] + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi}A_\phi$$

$$\nabla \times \mathbf{A} = \hat{r}\frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial \theta}\left(\sin(\theta)A_\phi\right) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\theta}\frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla^2 w = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial w}{\partial r}\right] + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left[\sin(\theta)\frac{\partial w}{\partial \theta}\right] + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 w}{\partial \phi^2}$$

Vector Identities

$A^2 = \mathbf{A}\cdot\mathbf{A}$

$\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$

$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$

$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$

$(\mathbf{A}+\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{C}+\mathbf{B}\cdot\mathbf{C}$

$(\mathbf{A}+\mathbf{B})\times\mathbf{C} = \mathbf{A}\times\mathbf{C}+\mathbf{B}\times\mathbf{C}$

$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B}$

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

$\nabla(v+w) = \nabla v + \nabla w$

$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$

$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$

$\nabla(vw) = v\nabla w + w\nabla v$

$\nabla\left(\frac{w}{v}\right) = \frac{v\nabla w - w\nabla v}{v^2}$

$\nabla w^n = nw^{n-1}\nabla w$

$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times\nabla\times\mathbf{B} + \mathbf{B}\times\nabla\times\mathbf{A}$

$\nabla\cdot(w\mathbf{A}) = w\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla w$

$\nabla\times(w\mathbf{A}) = w\nabla\times\mathbf{A} - \mathbf{A}\times\nabla w$

$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}$

$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

$\nabla\cdot\nabla w = \nabla^2 w$

$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$

$\nabla\times(v\nabla w) = \nabla v\times\nabla w$

$\nabla\times\nabla w = 0$

$\nabla\cdot\nabla\times\mathbf{A} = 0$

Metric Coefficients, Surface Elements, and Volume Elements for Different Systems

Relations	Cartesian (x, y, z)	Cylindrical (ρ, ϕ, z)	Spherical (r, θ, ϕ)
Base Vectors	$\hat{x}$ $\hat{y}$ $\hat{z}$	$\hat{\rho}$ $\hat{\phi}$ $\hat{z}$	$\hat{r}$ $\hat{\theta}$ $\hat{\phi}$
Metric Coefficients	1 1 1	1 ρ 1	1 r $r\sin(\theta)$
Surface Elements	$dS_x = dydz$ $dS_y = dxdz$ $dS_z = dxdy$	$dS_\rho = \rho d\phi dz$ $dS_\phi = d\rho dz$ $dS_z = \rho d\rho d\phi$	$dS_r = r^2 \sin(\theta) d\theta d\phi$ $dS_\theta = r\sin(\theta) drd\phi$ $dS_\phi = rdrd\theta$
Differential Volume	$dx\,dy\,dz$	$\rho\,d\rho\,d\phi\,dz$	$r^2\ \sin(\theta)\,dr\,d\theta\,d\phi$

Integral Identities

$$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left(x + \sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{dx}{\left(x^2 \pm a^2\right)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$$

$$\int \frac{xdx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$$

$$\int \frac{xdx}{\left(x^2 \pm a^2\right)^{3/2}} = -\frac{1}{\sqrt{x^2 \pm a^2}}$$

$$\int \begin{Bmatrix} \cos^2(x) \\ \sin^2(x) \end{Bmatrix} dx = \frac{x}{2} \pm \frac{\sin(2x)}{4}$$

$$\int \sin^3(x)dx = -\frac{1}{3}\left[2 + \sin^2(x)\right]\cos(x)$$

$$\int \cos^m(x)\sin(x)dx = -\frac{\cos^{m+1}(x)}{m+1}$$

$$\int_{-\pi}^{\pi} \cos(m\phi)\sin(n\phi)d\phi = 0$$

$$\int_{-\pi}^{\pi} \cos(m\phi)\cos(n\phi)d\phi = \int_{-\pi}^{\pi} \sin(m\phi)\sin(n\phi)d\phi = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$