ECE 435 Kempel

Homework Set #2

Fall 2000

Due: 9/18/00

All homework must be completed NEATLY!!! (If I can't read it, I can't grade it!)

All problems in Balanis, Antenna Theory, 2nd Edition

1. 2.4
2. 2.6
3. 2.11
4. 2.16 (Directiv.for is on the floppy that came with your book).
5. 2.22
6. 2.25
7. 2.28
8. 2.31

2.4 $U = B_0 \cos^3\theta$

(a)

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U \sin\theta \, d\theta \, d\phi = B_0 \int_0^{2\pi}\int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta \, d\phi$$

$$= B_0 \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta$$

$$= 2\pi B_0 \int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta = 2\pi B_0 \left[-\frac{\cos^4\theta}{4} \right]_0^{\pi/2}$$

$$= \frac{\pi}{2} B_0 = 10 \quad \therefore \quad B_0 = \frac{20}{\pi} = 6.3662$$

so

$$U = 6.3662 \cos^3\theta$$

$$W = \frac{U}{r^2} = \frac{6.3662 \cos^3\theta}{r^2} \quad @ \quad r = 10^3$$

$$= 6.3662 \times 10^{-6} \cos^3\theta$$

$$W_{max} = 6.3662 \times 10^{-6} \; W/m^2 \quad @ \quad \theta = 0^\circ$$

(b) $D_0 = \dfrac{4\pi U_{max}}{P_{rad}} = \dfrac{4\pi(6.3662)}{10} = 8$ or $9 \, dB$

(c) $G_0 = \varepsilon_t D_0 = 8 = 9 \, dB \qquad \varepsilon_t = 1$

2.6

(a) $D_0 \sim \dfrac{41{,}253}{\Theta_{1d}\Theta_{2d}} = \dfrac{41{,}253}{(30)(35)} = 39.29$ or $16 \, dB$

$$A_{em} = \frac{\lambda^2}{4\pi} D_0$$

(b) $D_0 \sim \dfrac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \dfrac{72{,}815}{(30)^2 + (35)^2} = 34.27$ or $15.4 \, dB$

2.11

(a) $P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{2\pi} \sin^2\phi\, d\phi \int_0^{\pi/2} \cos^4\theta \sin\theta\, d\theta$

in general — half-space

$= \pi\left(\frac{1}{5}\right) = \frac{\pi}{5}$

$U_{Max} = U(\theta = 0°, \phi = \pi/2°) = 1$

$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi}{\pi/5} = 20$ or 13 dB

(b) Elevation Plane $\phi = \text{const.} \Rightarrow \phi = \pi/2$

$U = \cos^4\theta \qquad 0 \le \theta \le \pi/2$

$\cos^4\left[\frac{HPBW}{2}\right] = 1/2 \Rightarrow HPBW = 2\cos^{-1}\left(\sqrt{0.5}\right) = 65.5°$

2.16

(a) $U = J_1^2(ka\sin\theta)$

$\rightarrow a = \frac{\lambda}{10} \Rightarrow ka\sin\theta = \frac{\pi}{5}\sin\theta \qquad HPBW = 93.10°$

$\rightarrow D_0 = \frac{101}{(93.1) - 0.0027(93.1)^2} = 1.45$

$\rightarrow D_0 = -172.4 + 191\sqrt{0.818 + 1/93.1} = 1.48$

$\rightarrow a = \frac{\lambda}{20} \Rightarrow ka\sin\theta = \frac{\pi}{10}\sin\theta \qquad HPBW = 91.10°$

$D_0 = 1.47$

$D_0 = 1.5$

b) $a = \lambda/10 \Rightarrow P_{rad} = 0.764$

$U_{Max} = 0.0893 \qquad D_0 = \frac{4\pi(0.0893)}{0.764} = 1.47$

$a = \lambda/20 \Rightarrow P_{rad} = 0.203$

$U_{Max} = 0.024 \qquad D_0 = 1.49$

as $a \to 0$, $D_0 \to 1.5$! (think infinitesimal dipole)

2.22

(a) Directiv gives $D_o = 14.0707 \Rightarrow 11.5\,dB$

(b) $U_{max} = 1$ @ $\theta = 0°$ $\quad U = \frac{\sin(\pi \sin\theta)}{\pi \sin\theta}$

Let $U = 1/2$ for HPBW

$\theta_1 = 26.3°$ so $HPBW = 2\theta_1 = 52.6° = \Theta_{1d}$

since it is omnidirectional $\Theta_{2d} = \Theta_{1d}$

$$D_o = \frac{41{,}253}{\Theta_{1d}\Theta_{2d}} = 14.91 \text{ or } 11.7\,dB$$

(c)

$$D_o = \frac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = 13.66 \text{ or } 11.2\,dB$$

2.25

(a) Lin-pol $\Delta\phi = 0$

(b) Lin-pol $\Delta\phi = \phi$

(c) circular $E_x = E_y$ $\quad \Delta\phi = \pi/2 \Rightarrow$ CCW since E_y leads E_x

$AR \equiv 1 \quad \tau = 90°$

(d) Circ-pol $\quad \Delta\phi = -\pi/2 \Rightarrow$ CW $\quad E_y$ lags E_x

$AR \equiv 1 \quad \tau = 90°$

(e) elliptical $\quad \Delta\phi \neq n\pi/2 \quad$ CCW (E_y leads E_x)

$OA = E\left[(1+1+\sqrt{2})/2\right]^{1/2} = 1.306\,E$

$OB = E\left[(1+1-\sqrt{2})/2\right]^{1/2} = 0.5412E$

$\Rightarrow AR = \frac{1.306}{0.5412} = \underline{2.4}$

$$\tau = 90° - \frac{1}{2}\tan^{-1}\left[\frac{2(1)\cos 45°}{1-1}\right] = 90° - \frac{1}{2}\tan^{-1}\left(\frac{1.414}{\phi}\right)$$

$$= 90° - \frac{1}{2}\,90° = 45°$$

(f) elliptical $\quad \Delta\phi \neq n\pi/2 \quad$ CW (E_y lags E_x)

$\left.\begin{array}{l} AR = 2.4 \\ \tau = 45° \end{array}\right\}$ as above

2.25 (con't)

(g) elliptical $E_x \neq E_y$ CCW (E_y leads E_x)

$$OA = E_y$$
$$OB = 0.5 E_y \Rightarrow AR = \frac{1}{0.5} = 2$$

$$\tau = 90^\circ - \frac{1}{2}\tan^{-1}\left(\frac{\phi}{-0.75}\right) = 90^\circ - \frac{1}{2}(180^\circ) = \phi^\circ$$

(h) elliptical CW (E_y lags E_x)

As above (g)

2.28

(a) $\bar{E}_w = \hat{x} E e^{-jkz} \Rightarrow \hat{\rho}_w = \hat{x}$

$E_a = (\hat{\theta} - j\hat{\phi}) E f(r,\theta,\phi) \Rightarrow \hat{\rho}_a = \frac{\hat{\theta} - j\hat{\phi}}{\sqrt{2}}$

$$PLF = |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \frac{1}{2} |\hat{x}\cdot\hat{\theta} - j\hat{x}\cdot\hat{\phi}|^2$$

$$\boxed{PLF = \frac{1}{2}\left[\cos^2\theta\cos^2\phi + \sin^2\phi\right]}$$

(b) $\hat{\rho}_a = \frac{\hat{\theta} + j\hat{\phi}}{\sqrt{2}} \Rightarrow$ same PLF !

2.31 (a) $\hat{\rho}_w = \frac{\hat{x} - j\hat{y}}{\sqrt{2}} \Rightarrow$ RH pol

$$\hat{\rho}_a = \frac{2\hat{x} \pm j\hat{y}}{\sqrt{5}}$$

$$PLF = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \begin{cases} \frac{9}{10} & (+) \\ \frac{1}{10} & (-) \end{cases}$$

(+) Ant LH X-mit, RH receive

(−) Ant RH X-mit, LH receive

(b) $\hat{\rho}_w = \frac{(\hat{x} + j\hat{y})}{\sqrt{2}} \Rightarrow$ LH pol

$$PLF = \begin{cases} \frac{1}{10} & (+) \\ \frac{9}{10} & (-) \end{cases}$$

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Homework Set #3

Fall 2000

Due: 9/25/00

All homework must be completed NEATLY!!! (If I can't read it, I can't grade it!)

All problems in Balanis, Antenna Theory, 2nd Edition

1. 2.63
2. 2.66
3. 2.67
4. 3.1
5. 3.2
6. 3.4

2-63. $\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 \cdot G_{ot} \cdot G_{or}$, $\lambda = \frac{3\times10^8}{9\times10^9} = \frac{3\times10^8}{90\times10^8} = \frac{1}{30}$

$R = 10,000$ meter $= \frac{10,000}{1/30}\lambda = 3\times10^5\lambda$.

$$\frac{P_r}{P_t} = \left[\frac{\lambda}{4\pi(3\times10^5\lambda)}\right]^2 \cdot G_o^2 = \frac{10\times10^{-6}}{10} = 10^{-6}$$

$$G_o^2 = 10^{-6}(4\pi\times3\times10^5)^2$$

$$G_o = 10^{-3}(4\pi\times3\times10^5) = 12\pi\times10^2 = 1200\pi$$

$$G_o = 1200\pi = 3,769.91 = 10\log_{10}(3,769.91)\ \text{dB}$$

$$G_o = 3,769.91 = 35.76\ \text{dB}$$

2-66. $\frac{P_r}{P_t} = \sigma \cdot \frac{G_{or}\cdot G_{ot}}{4\pi}\cdot\left[\frac{\lambda}{4\pi R_1 R_2}\right]^2 \Rightarrow \sigma = \frac{P_r\cdot 4\pi}{P_t\cdot G_{or}\cdot G_{ot}}\left[\frac{4\pi R_1\cdot R_2}{\lambda}\right]^2$

$$\lambda = \frac{3\times10^8}{3\times10^8} = 1\,\text{m}$$

$$\therefore \sigma = \frac{0.1425\times10^{-3}(4\pi)}{1000\,(75)(75)}\left[\frac{4\pi(500)(500)}{1}\right]^2 = 3142\ \text{m}^2 = \sigma$$

2-67.

$$\sigma = \frac{P_r\cdot 4\pi}{P_t\cdot G_{or}\cdot G_{ot}}\left[\frac{4\pi R_1 R_2}{\lambda}\right]^2$$

$$\lambda = \frac{3\times10^8}{1\times10^8} = 3\,\text{m}$$

$$\sigma = \frac{0.01\cdot(4\pi)}{1000\,(75)(75)}\left[\frac{4\pi(700)(700)}{3}\right]^2 = 94,114.5\ \text{m}^2$$

$$\sigma = 94,114.5\ \text{m}^2$$

3-1. If $\underline{H}_e = j\omega\epsilon \nabla \times \underline{\Pi}_e$ (1)

Maxwell's curl equation $\nabla \times \underline{E}_e = -j\omega\mu \underline{H}_e$ can be written as

$$\nabla \times \underline{E}_e = -j\omega\mu \underline{H}_e = -j\omega\mu(j\omega\epsilon \nabla \times \underline{\Pi}_e) = \omega^2\mu\epsilon \nabla \times \underline{\Pi}_e$$

or $\nabla \times (\underline{E}_e - \omega^2\mu\epsilon \underline{\Pi}_e) = \nabla \times (\underline{E}_e - k^2 \underline{\Pi}_e) = 0$ where $k^2 = \omega^2\mu\epsilon$

Letting

$$\underline{E}_e - k^2\underline{\Pi}_e = -\nabla\phi_e \Rightarrow \underline{E}_e = -\nabla\phi_e + k^2\underline{\Pi}_e \qquad (2)$$

Taking the curl of (1) and using the vector identity of Equation (3-8) leads to

$$\nabla \times \underline{H}_e = j\omega\epsilon \nabla \times \nabla \times \underline{\Pi}_e = j\omega\epsilon[\nabla(\nabla \cdot \underline{\Pi}_e) - \nabla^2\underline{\Pi}_e] \qquad (3)$$

Using Maxwell's equation

$$\nabla \times \underline{H}_e = \underline{J} + j\omega\epsilon \underline{E}_e$$

reduces (3) to

$$\underline{J} + j\omega\epsilon\underline{E}_e = j\omega\epsilon[\nabla(\nabla \cdot \underline{\Pi}_e) - \nabla^2\underline{\Pi}_e] \qquad (4)$$

Substituting (2) into (4) reduces to

$$\nabla^2\underline{\Pi}_e + k^2\underline{\Pi}_e = j\frac{\underline{J}}{\omega\epsilon} + [\nabla(\nabla \cdot \underline{\Pi}_e) + \nabla\phi_e] \qquad (5)$$

Letting $\phi_e = -\nabla \cdot \underline{\Pi}_e$ simplifies (5) to

$$\nabla^2\underline{\Pi}_e + k^2\underline{\Pi}_e = j\frac{\underline{J}}{\omega\epsilon} \qquad (6)$$

and (2) to

$$\underline{E}_e = \nabla(\nabla \cdot \underline{\Pi}_e) + k^2\underline{\Pi}_e \qquad (7)$$

Comparing (6) with (3-14) leads to the relation

$$\underline{\Pi}_e = -j\frac{1}{\omega\mu\epsilon}\underline{A} \qquad (8)$$

3-2. If $\underline{E}_m = -j\omega\mu \nabla \times \underline{\Pi}_m$ (1)

Maxwell's curl equation $\nabla \times \underline{H}_m = j\omega\epsilon \underline{E}_m$ can be written as

$$\nabla \times \underline{H}_m = j\omega\epsilon(-j\omega\mu \nabla \times \underline{\Pi}_m) = \omega^2\mu\epsilon \nabla \times \underline{\Pi}_m$$

or $\nabla \times (\underline{H}_m - \omega^2\mu\epsilon\underline{\Pi}_m) = \nabla \times (\underline{H}_m - k^2\underline{\Pi}_m) = 0$ where $k^2 = \omega^2\mu\epsilon$

Letting

$$\underline{H}_m - k^2\underline{\Pi}_m = -\nabla\phi_m \Rightarrow \underline{H}_m = -\nabla\phi_m + k^2\underline{\Pi}_m \qquad (2)$$

(Continued)

3-2 (Cont'd)

Taking the curl of (1) and using the vector identity of Equation (3-8) leads to

$$\nabla \times \underline{E}_m = -j\omega\mu \nabla \times \nabla \times \underline{\Pi}_m = -j\omega\mu [\nabla(\nabla \cdot \underline{\Pi}_m) - \nabla^2 \underline{\Pi}_m] \qquad (3)$$

Using Maxwell's equation

$\nabla \times \underline{E}_m = -\underline{M} - j\omega\mu \underline{H}_m$ reduces (3) to

$$-M - j\omega\mu \underline{H}_m = -j\omega\mu [\nabla(\nabla \cdot \underline{\Pi}_m) - \nabla^2 \underline{\Pi}_m] \qquad (4)$$

Substituting (2) into (4) reduces to

$$\nabla^2 \underline{\Pi}_m + k^2 \underline{\Pi}_m = j \frac{M}{\omega\mu} + [\nabla(\nabla \cdot \underline{\Pi}_m) + \nabla \phi_m] \qquad (5)$$

Letting $\phi_m = -\nabla \cdot \underline{\Pi}_m$ simplifies (5) to

$$\nabla^2 \underline{\Pi}_m + k^2 \underline{\Pi}_m = j \frac{M}{\omega\mu} \qquad (6)$$

and (2) to

$$\underline{H}_m = \nabla(\nabla \cdot \underline{\Pi}_m) + k^2 \underline{\Pi}_m \qquad (7)$$

Comparing (6) with (3-25) leads to the relation

$$\underline{\Pi}_m = -j \frac{1}{\omega\mu\epsilon} \underline{F}$$

3-4. The solution of $\nabla^2 A_z = -\mu J_z$ can be inferred from the solution of Poisson's equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon} \tag{1}$$

for the potential ϕ. $\rho(x', y', z')$ represents the charge density

We begin with Green's theorem

$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, dv' = \oint_\Sigma (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n}\, da \tag{2}$$

where ψ and ϕ are well behaved functions (nonsingular, continuous, and twice differentiable). For ψ we select a solution of the form

$$\psi = \frac{1}{R} \tag{3}$$

where $$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \tag{3a}$$

By considering the charge at the origin of the coordinate system, it can be shown that (provided $r \neq 0$)

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = \nabla^2\left(\frac{1}{r}\right) = 0$$

Thus (2) reduces to

$$\int_V \psi \nabla^2 \phi\, dv' = -\frac{1}{\epsilon}\int_V \frac{\rho(x', y', z')}{r}\, dv' \tag{4}$$

To exclude the $r=0$ singularity of ψ, the observation point x', y', z' is surrounded by a sphere of radius r' and surface Σ'. Therefore the volume V is bounded by the surfaces Σ and Σ', and (2) is broken into two integrals; one over Σ and the other of Σ'. Using (4) reduces (2) to

$$-\frac{1}{\epsilon}\int_V \frac{\rho}{r}\, dv' = \oint_\Sigma (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n}\, da + \oint_{\Sigma_0} (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n}\, da. \tag{5}$$

and

$$\oint_{\Sigma'} (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n}\, da = \oint_{\Sigma'} \left[\frac{1}{r'} \nabla \phi - \phi (\nabla \psi)_{r=r'}\right] \cdot \hat{n}\, da$$

$$= -\frac{1}{r'}\oint_{\Sigma'} \frac{\partial \phi}{\partial r}\, da - \frac{1}{r'^2}\oint_{\Sigma'} \phi\, da \tag{5a}$$

(Continued)

3-4 (Cont'd) Since r' is arbitrary, it can be chosen small enough so that ϕ and $\frac{\partial \phi}{\partial r}$ are essentially constant at every point on Σ'. If we make r' progressively smaller, ϕ and its normal derivative approach their limiting values at the center (by hypothesis, both exist and are continuous functions of position). Therefore in the limit as $r' \to 0$, both can be taken outside the integral and we can write that

$$\oint_{\Sigma'} (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n}\, da = -4\pi \phi(x,y,z) \qquad (6)$$

Since

$$\lim_{r' \to 0} \frac{1}{r'} \oint_{\Sigma'} \frac{\partial \phi}{\partial r} da = \lim_{r' \to 0} \frac{1}{r'} \left(\frac{\partial \phi}{\partial r}\right)_{r=r'} \oint_{\Sigma'} da = \lim_{r' \to 0} \frac{1}{r'} \left(\frac{\partial \phi}{\partial r}\right)_{r=r'} (4\pi r'^2) = 0$$

Substituting (6) into (5) reduces it to

$$\phi(x,y,z) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho}{r} dv' + \frac{1}{4\pi} \oint_\Sigma \left[\frac{1}{r} \nabla \phi - \phi \nabla\left(\frac{1}{r}\right)\right] \cdot \hat{n}\, da \qquad (7)$$

The first term on the right side of (7) accounts for the contributions from the charges within Σ while the second term for those outside Σ. Expansion of Σ to include all charges makes the second term to vanish and to reduce (7) to

$$\phi(x,y,z) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho(x',y',z')}{r} dv' \qquad (8)$$

By comparing $\nabla^2 A_z = -\mu J_z$ with (1) we can write that

$$A_z(x,y,z) = \frac{\mu}{4\pi} \int_V \frac{J_z(x',y',z')}{r} dv' \qquad (9)$$

For more details see D.T. Paris and F. K. Hurd, Basic Electromagnetic Theory, McGraw-Hill, 1969, pp. 128-131.

For the details of the solution of (3-31) see

R.E. Collin, Field Theory of Guided Waves, McGraw-Hill, 1960, pp. 35-39. It can be shown that

$$A_z = \frac{\mu}{4\pi} \int_V J_z(x',y',z') \frac{e^{-jkr}}{r} dv$$

Because of the length of the derivation, it will not be repeated here.

ECE 435 Kempel
Homework Set #4
Fall 2000

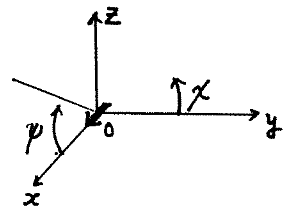
Due: 10/11/00

All homework must be completed NEATLY!!! (If I can't read it, I can't grade it!)

All problems in Balanis, Antenna Theory, 2nd Edition

1. 4.1
2. 4.7
3. 4.15
4. 4.16
5. 4.22
6. 4.28

4-1. a.



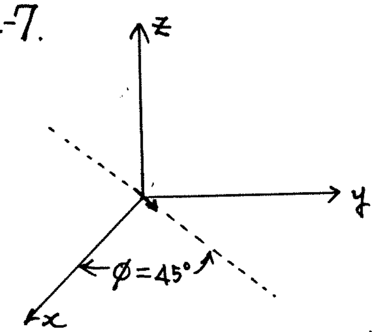
$$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_x\cdot\hat{a}_r|^2}$$
$$= \sqrt{1-(\sin\theta\cdot\cos\phi)^2}$$

In far-zone fields

$$E_\psi \simeq j\eta\frac{kI_0\cdot l e^{-jkr}}{4\pi r}\cdot\sin\psi = j\eta\frac{k\cdot I_0 l e^{-jkr}}{4\pi r}\cdot\sqrt{1-(\sin\theta\cdot\cos\phi)^2}$$
$$H_\chi \simeq j\frac{kI_0 l e^{-jkr}}{4\pi r}\cdot\sin\psi = \frac{E_\psi}{\eta}$$

b. $$U = U_0(1-\sin^2\theta\cos^2\phi)$$
$$\therefore P_{rad} = U_0\int_0^{2\pi}\int_0^{\pi}(1-\sin^2\theta\cdot\cos^2\phi)\cdot\sin\theta\, d\theta\, d\phi = U_0\cdot\frac{8\pi}{3}$$
$$D_0 = \frac{4\pi\cdot U_0}{U_0\cdot\frac{8\pi}{3}} = \frac{3}{2} = 1.5$$

4-7.



$$E_\psi = j\eta\frac{kI_0 l}{4\pi r}e^{-jkr}\sin\psi$$
$$H_\chi = j\frac{kI_0 l}{4\pi r}e^{-jkr}\sin\psi$$

Convert ψ to spherical coordinates

$$\sin\psi = 1-\cos^2\psi = \sqrt{1-\left(\frac{\hat{a}_x+\hat{a}_y}{\sqrt{2}}\cdot\hat{a}_r\right)^2}$$

$$\leftarrow \frac{\hat{a}_x+\hat{a}_y}{\sqrt{2}}\cdot\hat{a}_r = \left(\frac{\hat{a}_x}{\sqrt{2}}+\frac{\hat{a}_y}{\sqrt{2}}\right)\cdot(\hat{a}_x\sin\theta\cos\phi+\hat{a}_y\sin\theta\sin\phi+\hat{a}_z\cos\theta)$$
$$= \frac{\sin\theta\cos\phi}{\sqrt{2}}+\frac{\sin\theta\sin\phi}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin\theta(\cos\phi+\sin\phi)$$

Thus

$$E_\psi = j\eta\frac{kI_0 l}{4\pi r}e^{-jkr}\sqrt{1-\tfrac{1}{2}[\sin^2\theta(\cos\phi+\sin\phi)^2]}$$
$$H_\chi = j\frac{kI_0 l}{4\pi r}e^{-jkr}\sqrt{1-\tfrac{1}{2}[\sin^2\theta(\cos\phi+\sin\phi)^2]}$$

4-15. $\underline{A} = \hat{a}_z \frac{\mu I_0}{4\pi} \int_0^{l} e^{-jkz'} \frac{e^{-jkR}}{R} dz' \cong \hat{a}_z \frac{\mu I_0}{4\pi r} e^{-jkr} \int_0^{l} e^{-jk(1-\cos\theta)z'} dz'$

$$A_z \simeq \frac{\mu I_0 e^{-jkr}}{4\pi r} \int_0^{l} \frac{e^{-jk(1-\cos\theta)z'} \, d[-jk(1-\cos\theta)z']}{-jk(1-\cos\theta)}$$

$$A_z \simeq \frac{\mu I_0 e^{-jkr}}{4\pi r} \left[\frac{e^{-jk(1-\cos\theta)z'}}{-jk(1-\cos\theta)} \right]_0^{l} = \frac{\mu I_0 l e^{-jkr}}{4\pi r} e^{-jZ} \frac{\sin(Z)}{Z}$$

where $Z = \frac{kl}{2}(1-\cos\theta)$

(a) $\left.\begin{array}{l} A_r = A_z \cos\theta \\ A_\theta = -A_z \sin\theta \\ A_\phi = 0 \end{array}\right\} \Rightarrow$ For far-field $\Rightarrow \left\{\begin{array}{l} E_\theta \simeq -j\omega A_\theta \\ E_\phi \simeq -j\omega A_\phi \\ E_r \simeq 0 \end{array}\right.$

Therefore $E_r \simeq 0 \simeq H_r$, $E_\theta \simeq j \frac{\omega\mu I_0 l e^{-jkr}}{4\pi r} e^{-jZ} \frac{\sin(Z)}{Z} \sin\theta$

$E_\phi = 0 = H_\theta$, $H_\phi \simeq \frac{E_\theta}{\eta}$

(b) $\underline{W}_{ave} = \underline{W}_{rad} = \frac{1}{2} \mathrm{Re}[\underline{E} \times \underline{H}^*] = \frac{1}{2\eta} |E_\theta|^2$

$$= \frac{1}{2\eta} \left| \frac{\omega\mu I_0 l}{4\pi r} \cdot \frac{\sin(Z)}{Z} \cdot \sin\theta \right|^2$$

4-16. (a) $\underline{A} = \frac{\mu}{4\pi}\int_{-\infty}^{\infty} \underline{I}(z') \frac{e^{-jkR}}{R} dz' = \hat{a}_z \frac{\mu I_0}{4\pi}\int_{-\infty}^{\infty} \frac{e^{-jkR}}{R} dz'$

where $R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\Big|_{x'=y'=0} = \sqrt{x^2+y^2+(z-z')^2}$

Making a change of variable of the form,

$z - z' = -P \;,\; dz' = dP$

reduces the potential to

$$A_z = \frac{\mu I_0}{4\pi}\int_{-\infty}^{\infty} \frac{e^{-jk\sqrt{\rho^2+P^2}}}{\sqrt{\rho^2+P^2}} dP \qquad \text{where } \rho^2 = x^2+y^2$$

Using $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-j\beta\sqrt{b^2+t^2}}}{\sqrt{b^2+t^2}} dt = -j\pi H_0^{(2)}(b\beta)$

We can write the potential as

$$A_z = -j\frac{\mu I_0}{4} H_0^{(2)}(k\rho) = -j\frac{\mu I_0}{4} H_0^{(2)}(k\sqrt{x^2+y^2})$$

(b) $\underline{H} = \frac{1}{\mu}\nabla\times\underline{A}$ and $\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H}$

Since $A_\rho = A_\phi = 0$, in cylindrical coordinates

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} = \frac{1}{\mu}\left(-\hat{a}_\phi \frac{\partial A_z}{\partial\rho}\right) = \hat{a}_\phi\, j\frac{I_0}{4}\frac{\partial}{\partial\rho} H_0^{(2)}(k\rho)$$

Using Equation (V-19), we can write the H-field as

$$\underline{H} = \hat{a}_\phi H_\phi = -\hat{a}_\phi\, j\frac{kI_0}{4} H_1^{(2)}(k\rho)$$

where $H_1^{(2)}(k\rho)$ is the Hankel function of the second kind of order one and argument $k\rho$.

The electric field can be obtained using

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} = \hat{a}_z \frac{1}{j\omega\epsilon}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\right] = \hat{a}_z\frac{1}{j\omega\epsilon}\left(\frac{\partial H_\phi}{\partial\rho} + \frac{H_\phi}{\rho}\right)$$

$$= \hat{a}_z\frac{1}{j\omega\epsilon}\left[-j\frac{kI_0}{4}\frac{\partial}{\partial\rho}H_1^{(2)}(k\rho) - j\frac{kI_0}{4\rho}H_1^{(2)}(k\rho)\right]$$

Since $\frac{\partial}{\partial\rho}H_1^{(2)}(k\rho) = kH_0^{(2)}(k\rho) - \frac{1}{\rho}H_1^{(2)}(k\rho)$ by using V-18

then

$$\underline{E} = \hat{a}_z\left[-\frac{kI_0}{4\omega\epsilon}kH_0^{(2)}(k\rho)\right] = -\hat{a}_z\,\eta\,\frac{I_0 k}{4}H_0^{(2)}(k\rho)$$

4-22. (a) $VSWR = \frac{1+|\Gamma|}{1-|\Gamma|} \Rightarrow |\Gamma| = \frac{VSWR-1}{VSWR+1} = \left|\frac{2-1}{2+1}\right| = \left|\frac{1}{3}\right|$

$$|\Gamma| = \left|\frac{1}{3}\right| = \left|\frac{Z_{in}-Z_c}{Z_{in}+Z_c}\right| = \left|\frac{Z_{in}/Z_c - 1}{Z_{in}/Z_c + 1}\right| = \begin{cases} \left|\frac{2-1}{2+1}\right| \Rightarrow \frac{Z_{in}}{Z_c} = 2 \\ \left|\frac{\frac{1}{2}-1}{\frac{1}{2}+1}\right| \Rightarrow \frac{Z_{in}}{Z_c} = \frac{1}{2} \end{cases}$$

Largest $\frac{Z_{in}}{Z_c} = 2 \Rightarrow Z_{in} = 2Z_c = 100$

(b) $R_{in} = 11.14\ G^{4.17}$ $\quad \lambda/2 < \ell < 2\lambda/\pi$

$\pi/2 < k\ell/2 < 2$

$100 = 11.14\ G^{4.17}$

$\frac{100}{11.14} = G^{4.17}$, $8.9767 = G^{4.17}$

$\log_{10}(8.9767) = 4.17 \log_{10}(G)$, $0.953 = 4.17 \log_{10} G$

$0.2286 = \log_{10} G$, $G = 10^{0.2286} = 1.6928 = \frac{k\ell}{2} = 96.99°$

$k\ell = 2(1.6928)$, $\ell = \frac{2(1.6928)\lambda}{2\pi} = \frac{1.6928}{\pi}\lambda = 0.5388\lambda$

$\ell = 0.5388\lambda$

(c) $R_{in} = \frac{R_r}{\sin^2(\frac{k\ell}{2})} \Rightarrow R_r = R_{in}\sin^2(\frac{k\ell}{2}) = 100\sin^2(96.99°)$

$R_{in} = 100(0.9926)^2 = 100(0.9852) = 98.52$ ohms

$R_{in} = 98.52$ ohms

4-28. $\ell = \lambda/2$, $Z_c = 50$ ohms

$Z_{in} = 73 + j42.5$, $Y_{in} = \frac{1}{Z_{in}} = \frac{1}{73+j42.5} \cdot \frac{73-j42.5}{73-j42.5}$

$Y_{in} = 0.01023 - j0.0059563 = (10.23 - j5.9563) \times 10^{-3} = G_{in} - jB_{in}$

$B_{in} = \omega C_{in} = 2\pi f C_{in} \Rightarrow C_{in} = \frac{B_{in}}{2\pi f} = \frac{5.9563 \times 10^{-3}}{2\pi \cdot (10 \times 10^8)} = 0.94797 \times 10^{-12}$

$\therefore C_{in} = 0.94797$ pF

$G_{in} = 10.23 \times 10^{-3}$

$R_{in} = \frac{1}{G_{in}} = 97.75$, $\Gamma_{in} = \frac{R_{in} - Z_c}{R_{in} + Z_c} = \frac{97.75 - 50}{97.75 + 50} = 0.3232$

$VSWR = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} = \frac{1 + 0.3232}{1 - 0.3232} = 1.955$

5-13. Since $E_\phi \sim J_1(ka\sin\theta)$

a. $E_\phi\big|_{\theta=0} = J_1(ka\sin\theta)\big|_{\theta=0} = J_1(0) = 0$

$E_\phi\big|_{\theta=\pi/2} = J_1(ka\sin\theta)\big|_{\theta=90^\circ} = J_1(ka) = 0 \Rightarrow ka = 3.84$

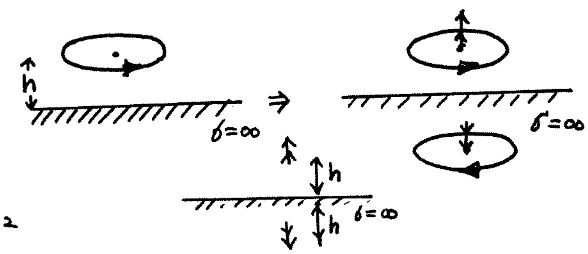
Thus $a = \frac{3.84}{k} = \frac{3.84\lambda}{2\pi} = 0.61115\lambda$

b. Since $a = 0.61115\lambda > 0.5\lambda$, use large loop approximation. According to (5-63a) $R_r = 60\pi^2(C/\lambda) = 60\pi^2\left(\frac{2\pi a}{\lambda}\right) = 60\pi^2(2\pi(0.61115))$
$= 2{,}273.94$

c. The directivity is given by (5-63b), or

$D_0 = 0.682\left(\frac{C}{\lambda}\right) = 0.682\left(\frac{2\pi a}{\lambda}\right) = 0.682(2\pi)(0.61115) = 2.619$

5-18.



a. $E_\phi = \eta \frac{(ka)^2 I_0 e^{-jkr}}{4r}\sin\theta$

$|AF| = |2j\sin(kh\cos\theta)|$

$E_\phi = \eta \frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta$, $S = \pi a^2$

$(E_\phi)_t = E_\phi(AF) = \eta\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta\,[2j\sin(kh\cos\theta)]$, ← above ground plane total field.

b. $h = \lambda$, $kh = 2\pi$

$\sin\theta\cdot[2j\sin(2\pi\cos\theta)] = 0$, $\sin(2\pi\cos\theta) = 0$, $2\pi\cos\theta = n\pi$, $n = 0, 1, 2$

$\theta_n = 0^\circ$, $\cos\theta_n = \frac{n}{2}$, $n = 0, 1, 2 \rightarrow \theta_n = 90^\circ$,

$\theta_0 = \cos^{-1}(0) = 90^\circ$

$\theta_1 = \cos^{-1}(\tfrac{1}{2}) = 60^\circ$

$\theta_2 = \cos^{-1}(1) = 0^\circ$.

(Continued)

5-18. (Cont'd)

c. $(E_\phi)_t = C\sin\theta\sin(kh\cos\theta)\big|_{\theta=60^\circ} = 0 = C\cdot\left(\frac{\sqrt{3}}{2}\right)\cdot\sin\left(\frac{2\pi}{\lambda}\cdot h\cdot\frac{1}{2}\right) = C\cdot\frac{\sqrt{3}}{2}\cdot\sin\left(\frac{\pi h}{\lambda}\right)$

$\sin\left(\frac{\pi h}{\lambda}\right) = 0 \Rightarrow \frac{\pi h}{\lambda} = \sin^{-1}(0) = n\pi$, $n = 0, 1, 2, 3, \cdots$

$\frac{h}{\lambda} = \pm n \Rightarrow$ Physical Nonzero height $\Rightarrow h = n\lambda$, $n = 1, 2, 3, \cdots$