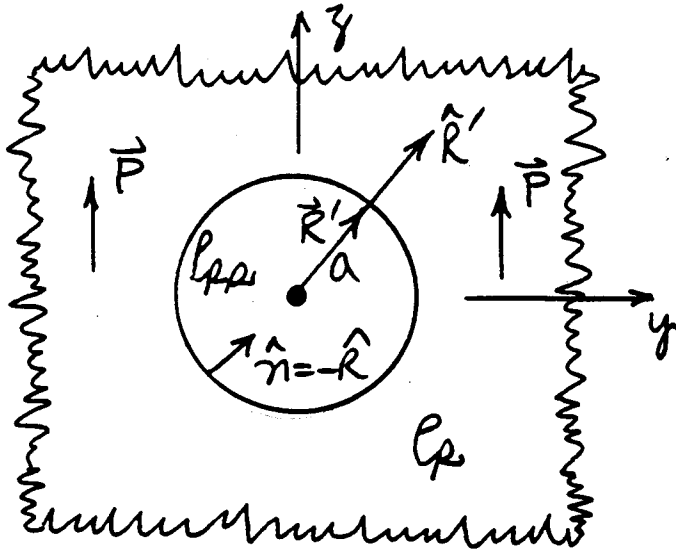


PROBLEM SET 5

P. 3-23. Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization \vec{P} exists.



Assume $\vec{P} = \hat{z} P_0 = \text{uniform}$

$$\rho_p = -\nabla \cdot \vec{P} = 0 \dots \text{for } \vec{P} = \text{const.}$$

$$\rho_{ps} = \hat{n} \cdot \vec{P} = -\hat{z} \cdot \hat{z} P_0 = -P_0 \text{ on surface}$$

$$\vec{R} = 0, \quad \vec{R}' = \hat{R}' a$$

$$\vec{R} - \vec{R}' = -\hat{R}' a, \quad |\vec{R} - \vec{R}'| = a$$

$$\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0} \int_S \rho_{ps}(\vec{R}') \frac{\vec{R} - \vec{R}'}{|\vec{R} - \vec{R}'|^3} ds'$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\pi -P_0 \cos\theta' \frac{-\hat{R}' a}{a^3} a^2 \sin\theta' d\theta' d\phi'$$

variable \swarrow

$$= \frac{P_0}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \cos\theta' \left[\hat{x} \sin\theta' \cos\phi' + \hat{y} \sin\theta' \sin\phi' + \hat{z} \cos\theta' \right] \times \sin\theta'$$

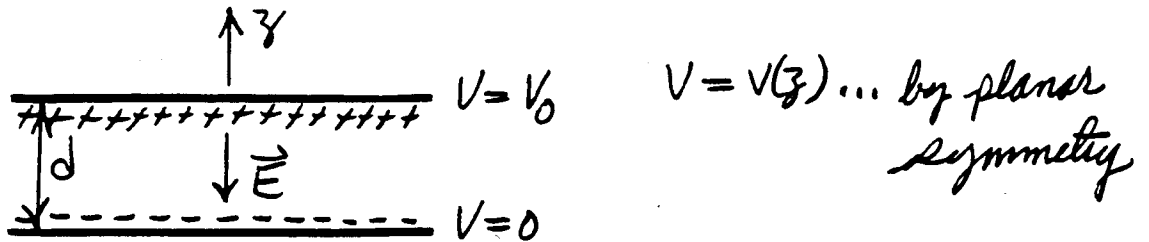
by integration

$$= \frac{\hat{z} P_0}{2\epsilon_0} \int_0^\pi \cos^2\theta' \sin\theta' d\theta' = \frac{\hat{z} P_0}{2\epsilon_0} \frac{-1}{3} \cos^3\theta' \Big|_0^\pi$$

$$\vec{E}(0) = \hat{z} \frac{P_0}{2\epsilon_0}$$

P.3-24 Solve the following problems:

- a) Find the breakdown voltage of a parallel-plate capacitor, assuming that conducting plates are 50 (mm) apart and the medium between them is air.



$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = 0 \quad \dots \text{in space region}$$

$$\therefore \vec{E} = -\nabla V = -\hat{z} \frac{\partial V}{\partial z} = \hat{z} E_z(z)$$

$$\frac{\partial E_z}{\partial z} = 0 \rightarrow E_z = C = \text{const.}$$

$$V_0 = -\int_0^d E_z dz = -C \int_0^d dz = -Cd \rightarrow C = -\frac{V_0}{d}$$

$$E_z = -\frac{V_0}{d}, \quad \therefore E_L = \frac{(V_0)_{\max}}{d}$$

$$(V_0)_{\max} = E_L d = \frac{3 \text{ kV}}{\text{mm}} \times 50 \text{ mm}$$

$$(V_0)_{\max} = 150 \text{ kV}$$

b) Find the breakdown voltage if the entire space between the conducting plates is filled with plexiglass, which has a dielectric constant 3 and a dielectric strength 20 (kV/mm).

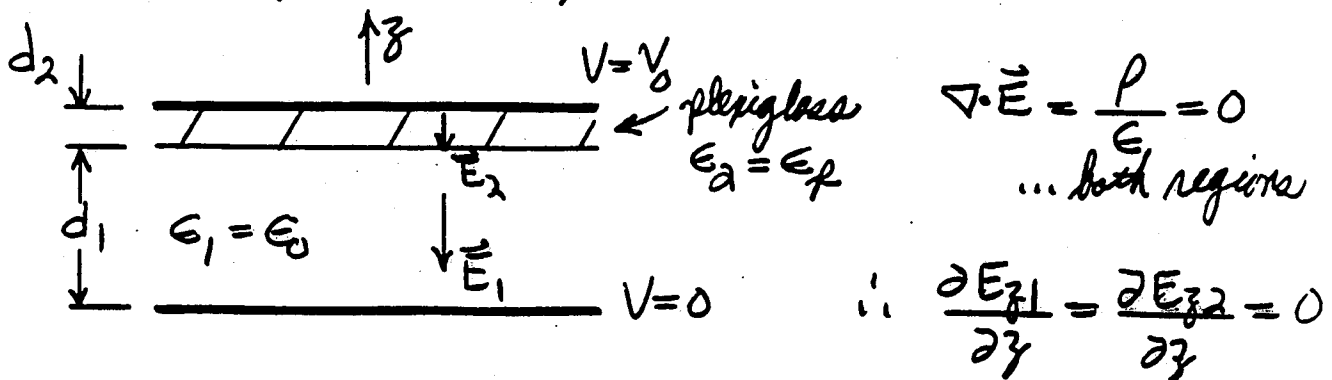
$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon} = 0 \dots \text{in capacitor, so analysis doesn't change and}$$

$$E_z = -\frac{V_0}{d}, \quad \therefore E_z = \frac{(V_0)_{\max}}{d}$$

$$(V_0)_{\max} = E_z d = 20 \frac{\text{kV}}{\text{mm}} \times 50 \text{ mm}$$

$$(V_0)_{\max} = 1000 \text{ kV}$$

c) If a 10 (mm) thick plexiglass is inserted between the plates, what is the maximum voltage that can be applied to the plates without a breakdown?



$\therefore E_{z1} = C_1, E_{z2} = C_2 \dots$ constants in both regions

$$V_0 = -\int_0^d E_z dz = -C_1 \int_0^{d_1} dz - C_2 \int_{d_1}^{d_1+d_2} dz$$

$$= -C_1 d_1 - C_2 d_2$$

$$\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \rho_{\text{free}} = 0 \rightarrow \hat{z} \cdot (\epsilon_r \vec{E}_2 - \epsilon_0 \vec{E}_1) = 0 \dots \text{at interface}$$

between plexiglass
and air

↑ B.C. for normal \vec{D}

$$\epsilon_r E_{z2} = \epsilon_0 E_{z1} \rightarrow \epsilon_r C_2 = \epsilon_0 C_1$$

$$C_2 = \frac{\epsilon_0}{\epsilon_r} C_1 = \frac{C_1}{\epsilon_{rr}}$$

$$V_0 = -\left(d_1 + \frac{d_2}{\epsilon_{rr}}\right) C_1 \rightarrow E_{z1} = -\frac{V_0}{d_1 + \frac{d_2}{\epsilon_{rr}}}$$

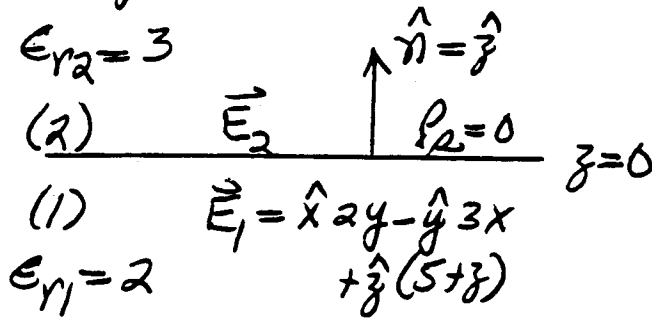
Note: $E_{z2} = \frac{E_{z1}}{\epsilon_{rr}} < E_{z1}$ and $E_{fr} > E_{fa}$

∴ air region breaks down first and

$$(V_0)_{\text{max}} = (E_b)_{\text{air}} \left(d_1 + \frac{d_2}{\epsilon_{rr}}\right) = 3 \frac{\text{kV}}{\text{mm}} \left(40 + \frac{10}{3}\right) \text{mm}$$

$$(V_0)_{\text{max}} = 130 \text{ kV}$$

P. 3-25 Assume that the $z=0$ plane separates two lossless dielectric regions with $\epsilon_{r1}=2$ and $\epsilon_{r2}=3$. If we know that \vec{E}_1 in region (1) is $\hat{x}2y - \hat{y}3x + \hat{z}(5+z)$, what do we also know about \vec{E}_2 and \vec{D}_2 in region 2? Can we determine \vec{E}_2 and \vec{D}_2 at any point in region 2? Explain.



Assume $\rho_2 = 0$

Use: $E_{x1} = E_{x2}$

$\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = 0$ at $z=0$

$$D = \epsilon E$$

$$E_{x1} = E_{x2} \rightarrow E_{x2} = E_{x1} = 2y, \quad E_{y2} = E_{y1} = -3x$$

$$\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = 0 \rightarrow \hat{z} \cdot (\epsilon_2 \vec{E}_2 - \epsilon_1 \vec{E}_1) = 0, \quad \epsilon_{r2} E_{z2} = \epsilon_{r1} E_{z1}$$

$$\text{or } E_{z2} = \frac{\epsilon_{r1}}{\epsilon_{r2}} E_{z1} = \frac{2}{3}(5)$$

$$\vec{E}_2 = \hat{x}2y - \hat{y}3x + \hat{z} \frac{10}{3}$$

$$\vec{D}_2 = \epsilon_2 \vec{E}_2 = 3\epsilon_0 \left(\hat{x}2y - \hat{y}3x + \hat{z} \frac{10}{3} \right)$$

... at $z=0$

(\vec{E}_2, \vec{D}_2) Unknown for $z > 0$, because boundary conditions only relate surface values.

$$V_0 = -\int_0^d E_z dz = -C_1 \int_0^{d_1} dz - C_2 \int_{d_1}^{d_1+d_2} dz$$

$$= -C_1 d_1 - C_2 d_2$$

$$\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \rho_{s1} = 0 \rightarrow \hat{z} \cdot (\epsilon_r \vec{E}_2 - \epsilon_0 \vec{E}_1) = 0 \dots \text{at interface}$$

between plexiglass
and air

↑ B.C. for normal \vec{D}

$$\epsilon_r E_{z2} = \epsilon_0 E_{z1} \rightarrow \epsilon_r C_2 = \epsilon_0 C_1$$

$$C_2 = \frac{\epsilon_0}{\epsilon_r} C_1 = \frac{C_1}{\epsilon_{rp}}$$

$$V_0 = -\left(d_1 + \frac{d_2}{\epsilon_{rp}}\right) C_1 \rightarrow E_{z1} = -\frac{V_0}{d_1 + \frac{d_2}{\epsilon_{rp}}}$$

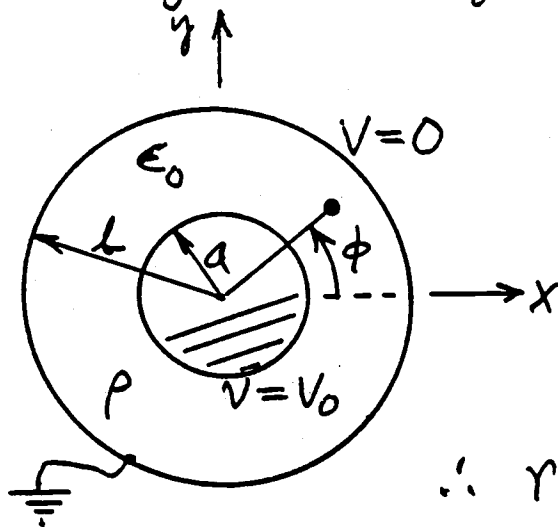
Note! $E_{z2} = \frac{E_{z1}}{\epsilon_{rp}} < E_{z1}$ and $E_{tp} > E_{ta}$

∴ air region breaks down first and

$$(V_0)_{\max} = (E_{t, \text{air}}) \left(d_1 + \frac{d_2}{\epsilon_{rp}}\right) = 3 \frac{\text{kV}}{\text{mm}} \left(40 + \frac{10}{3}\right) \text{mm}$$

$$(V_0)_{\max} = 130 \text{ kV}$$

P.4-6 Assume that the space between the inner and outer conductors of a long coaxial cylindrical structure is filled with an electron cloud having a volume density of charge $\rho = A/r$ for $a < r < b$, where a and b are the radii of the inner and outer conductors, respectively. The inner conductor is maintained at a potential V_0 , and the outer conductor is grounded. Determine the potential distribution in the region $a < r < b$ by solving Poisson's equation.



$$V(r, \phi, z) = V(r) \dots \text{by cylindrical symmetry}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \dots \text{Poisson's eq.}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) = -\frac{A}{\epsilon_0 r}$$

$$\therefore r \frac{\partial V}{\partial r} = -\frac{A}{\epsilon_0} r + C_1$$

$$\frac{\partial V}{\partial r} = -\frac{A}{\epsilon_0} + \frac{C_1}{r} \rightarrow V(r) = -\frac{A}{\epsilon_0} r + C_1 \ln r + C_2$$

$$V(r=b) = 0 \rightarrow C_2 = \frac{Ab}{\epsilon_0} - C_1 \ln b$$

$$V(r) = -\frac{A}{\epsilon_0} (r-b) + C_1 \ln \left(\frac{r}{b} \right)$$

$$V(r=a) = V_0 \rightarrow \frac{A(b-a)}{\epsilon_0} + C_1 \ln \left(\frac{a}{b} \right) = V_0$$

$$C_1 = \frac{V_0 - A(b-a)/\epsilon_0}{\ln(a/b)} = \frac{A(b-a)/\epsilon_0 - V_0}{\ln(b/a)}$$

$$\begin{aligned} C_2 &= \frac{Ab}{\epsilon_0} - C_1 \ln b = \frac{A/\epsilon_0 [b \ln(b/a) - (b-a) \ln b] + V_0 \ln b}{\ln(b/a)} \\ &= \frac{V_0 \ln b + A/\epsilon_0 (a \ln b - b \ln a)}{\ln(b/a)} \end{aligned}$$

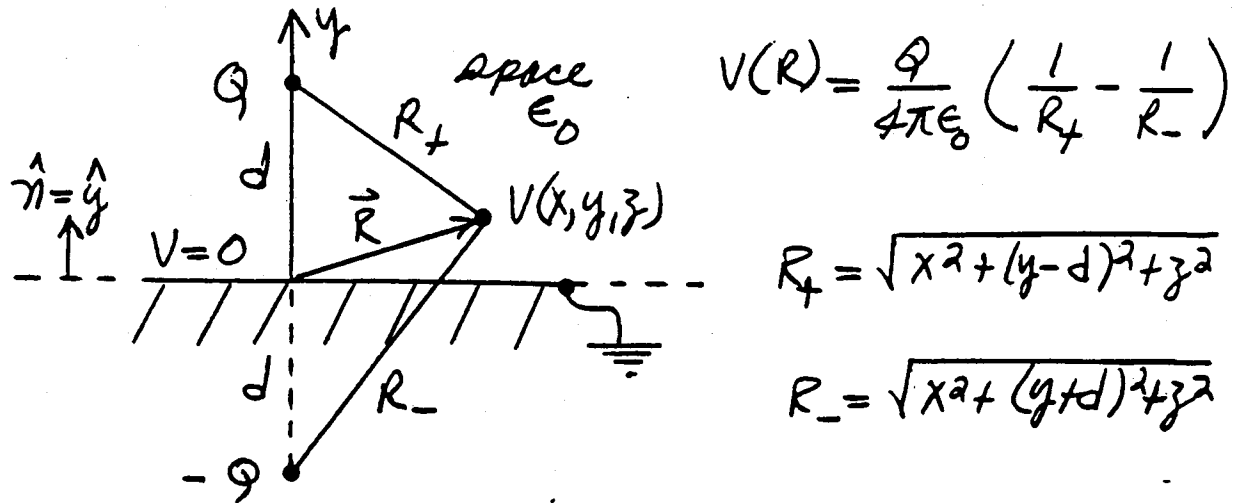
$$V(r) = -\frac{A}{\epsilon_0} r + C_1 \ln r + C_2$$

$$C_1 = \frac{A(b-a) - V_0}{\ln(b/a)}$$

$$C_2 = \frac{V_0 \ln b + A/\epsilon_0 (a \ln b - b \ln a)}{\ln(b/a)}$$

P.4-7 A point charge exists at a distance d above a large grounded conducting plane. Determine

a) the surface charge density ρ_s ,



$$\rho_s = \epsilon_0 (\hat{n} \cdot \vec{E}) = -\epsilon_0 (\hat{y} \cdot \nabla V) = -\epsilon_0 \frac{\partial V}{\partial y} \Big|_{y=0} \quad \dots \text{at conductor surface}$$

$$\begin{aligned} \rho_s &= -\epsilon_0 \frac{Q}{4\pi\epsilon_0} \frac{\partial}{\partial y} \left(R_+^{-1} - R_-^{-1} \right) \Big|_{y=0} \\ &= -\frac{Q}{4\pi} \left[-R_+^{-2} \frac{\partial R_+}{\partial y} + R_-^{-2} \frac{\partial R_-}{\partial y} \right] \Big|_{y=0} \\ &= -\frac{Q}{4\pi} \left[-\frac{1}{R_+^2} \frac{(y-d)}{R_+} + \frac{1}{R_-^2} \frac{(y+d)}{R_-} \right] \Big|_{y=0} \\ &= -\frac{Q}{4\pi} \left[\frac{(y+d)}{R_-^3} - \frac{(y-d)}{R_+^3} \right] \Big|_{y=0} \end{aligned}$$

$$\boxed{\rho_s = -\frac{2Qd}{4\pi(x^2+d^2+z^2)^{3/2}}}$$

b) the total charge induced on the conducting plane.

$$Q_{\text{ind}} = \int_S \rho_2 ds = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho_2(x, z) dx dz$$

Change to polar coordinates in the x - z plane, i.e., let $r^2 = x^2 + z^2$ and $ds = dx dz = r dr d\theta$, then

$$\begin{aligned} Q_{\text{ind}} &= \int_{-\pi}^{\pi} \int_0^{\infty} \rho_2(r) r dr d\theta \\ &= 2\pi \int_0^{\infty} \frac{-Qd}{2\pi (r^2 + d^2)^{3/2}} r dr \\ &= -Qd \left. \frac{-1}{\sqrt{r^2 + d^2}} \right|_{r=0}^{r=\infty} = -Q \end{aligned}$$

\therefore
$$Q_{\text{ind}} = -Q$$