Stability Analysis of Periodically Switched Linear Systems Using Floquet Theory

Cevat Gökçek

Department of Mechanical Engineering
Michigan State University
East Lansing, MI 48824
gokcek@me.msu.edu

Abstract

Stability of a switched system that consists of a set of linear time invariant subsystems and a periodic switching rule is investigated. Based on the Floquet theory, necessary and sufficient conditions are given for exponential stability. It is shown that there exists a slow switching rule that achieves exponential stability if at least one of these subsystems is asymptotically stable. It is also shown that there exists a fast switching rule that achieves exponential stability if the average of these subsystems is asymptotically stable. The results are illustrated by examples.

Keywords

Periodically switched linear systems, Floquet theory, exponential stability.

Please send all correspondence to Professor Cevat Gokcek, Department of Mechanical Engineering, Michigan State University, East Lansing, MI, 48824, USA, phone: +1 (517) 432-6438, fax: +1 (517) 353-1750, email: `gokcek@me.msu.edu`.

1. INTRODUCTION

A switched system consists of a set of subsystems and a rule that describes switching among them. The subsystems may be continuous time or discrete time systems and the switching rule may depend on time or states of individual subsystems.

Switched systems arise when dynamics of systems undergo abrupt changes due to component failures, parameter changes, switching elements or switching controllers. Such systems have been studied extensively in the literature. A recent survey of switched systems can be found in [1] and various applications of switched systems are discussed in [2].

In this paper, a special class of switched systems is considered. Specifically, we consider a periodically switched linear system of the form

$$\dot{x}(t) = \begin{cases} A_1 x(t), & t_0 + lT \le t < t_1 + lT, \\ A_2 x(t), & t_1 + lT \le t < t_2 + lT, \\ \qquad \vdots & \\ A_\sigma x(t), & t_{\sigma-1} + lT \le t < t_\sigma + lT, \end{cases} \qquad l = 0, 1, 2, \cdots, \quad t \ge t_0, \tag{1.1}$$

$$x(t_0) = x_0,$$

with $t_\sigma = t_0 + T$, where $x(t) \in \mathbb{R}^n$ and A_1, A_2, $\cdots$, $A_\sigma \in \mathbb{R}^{n \times n}$ are (not necessarily different) constant matrices. Defining

$$\Delta t_k = t_k - t_{k-1}, \quad k = 1, \cdots, \sigma, \tag{1.2}$$

it can be easily seen that each subsystem $\dot{x}(t) = A_k x(t)$ is active for Δt_k seconds within each period. Systems of this type can be used to model sampled data control systems, switch mode power supplies, switched capacitor filters and switching amplifiers.

The goal of this paper is to investigate stability of periodically switched linear systems of the above form. The main tool used for this purpose is the Floquet theory [3], [4]. It should be mentioned that although Floquet theory has been used in the literature to investigate stability of periodically time varying linear systems, there seems to be no explicit study of stability of periodically switched linear systems using this theory.

This paper is organized as follows. In Section 2, the Floquet theory is reviewed. In Section 3, the stability theorems are presented. In Section 4, the results are illustrated. In Section 5, conclusions are given.

2. FLOQUET THEORY

Floquet theory transforms a linear periodically time varying system into a linear time invariant system through a Lyapunov transformation. Hence, the stability of the former system can be inferred from that of the latter system. Below is a brief review of the Floquet theory.

Consider the linear time varying system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \;\; t \geq t_0, \\ x(t_0) &= x_0, \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, the matrix $A(t) \in \mathbb{R}^{n \times n}$ is piecewise continuous, bounded and periodic with period T. Letting $\Phi(t, t_0)$ be the state transition matrix of the system (2.1), the Floquet theorem [3], [4] states that

(a) $\Phi(t+T, t_0+T) = \Phi(t, t_0)$.

(b) There exist a nonsingular matrix $P(t, t_0)$, which satisfies $P(t+T, t_0) = P(t, t_0)$, and a constant matrix Q such that

$$\Phi(t, t_0) = P(t, t_0) \exp[Q(t - t_0)]. \tag{2.2}$$

(c) The Lyapunov transformation

$$z(t) = P^{-1}(t, t_0)x(t) \tag{2.3}$$

transforms the system (2.1) into the linear time invariant system

$$\begin{aligned} \dot{z}(t) &= Qz(t), \;\; t \geq t_0, \\ z(t_0) &= x_0. \end{aligned} \tag{2.4}$$

Letting

$$R = \Phi(t_0 + T, t_0), \tag{2.5}$$

the matrix Q can be expressed as

$$Q = \frac{1}{T} \log(R). \tag{2.6}$$

The eigenvalues λ_k, $k = 1, \cdots, n$, of the matrix Q are called the characteristic exponents and the eigenvalues μ_k, $k = 1, \cdots, n$, of the matrix R are called the characteristic multipliers. The characteristic exponents are related to the characteristic multipliers through

$$\mu_k = \exp(\lambda_k T), \;\; k = 1, \cdots, n. \tag{2.7}$$

Hence, it follows from the Floquet theorem that the system (2.1) is exponentially stable if and only if the matrix Q is Hurwitz, i.e., all eigenvalues of Q have negative real parts. Equivalently, the system (2.1) is exponentially stable if and only if the matrix R is Schur, i.e., all eigenvalues of R have magnitudes less than one.

Note that the matrix Q defined above is not necessarily real. If it turns out to be complex, then a real Q can be obtained from

$$Q = \frac{1}{2T} \log[\Phi(t_0 + 2T, t_0)] = \frac{1}{2T} \log(R^2). \tag{2.8}$$

This follows from the fact that $A(t)$ is also periodic with period $2T$.

3. STABILITY RESULTS

Consider the periodically switched linear system discussed above. Application of Floquet theory to this system yields the following result.

Theorem 3.1: The system (1.1) is exponentially stable if and only if the matrix

$$R = \prod_{k=1}^{\sigma} \exp(A_k \Delta t_k) = \exp(A_\sigma \Delta t_\sigma) \exp(A_{\sigma-1} \Delta t_{\sigma-1}) \cdots \exp(A_1 \Delta t_1) \tag{3.1}$$

is Schur. Equivalently, the system (1.1) is exponentially stable if and only if the matrix

$$Q = \frac{1}{T} \log[\prod_{k=1}^{\sigma} \exp(A_k \Delta t_k)] \tag{3.2}$$

is Hurwitz. ■

Proof: For each $k = 1, \cdots, \sigma$, define $p_k(t)$ as

$$p_k(t) = \begin{cases} 1, & t_{k-1} + lT \leq t < t_k + lT, \\ 0, & \text{otherwise}, \end{cases} \qquad l = 0, 1, 2, \cdots. \tag{3.3}$$

Letting

$$A(t) = A_1 p_1(t) + A_2 p_2(t) + \cdots + A_\sigma p_\sigma(t), \tag{3.4}$$

the system (1.1) can be compactly written as

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \ \ t \geq t_0, \\ x(t_0) &= x_0. \end{aligned} \tag{3.5}$$

Clearly, $A(t)$ is piecewise continuous, bounded and periodic with period T. Thus, the hypothesis of the Floquet theorem is satisfied.

For $t_0 \le t < t_1$, $\dot{x}(t) = A_1 x(t)$ so that $x(t) = \exp[A_1(t-t_0)]x(t_0) = \exp[A_1(t-t_0)]x_0$. Similarly, for $t_1 \le t < t_2$, $\dot{x}(t) = A_2 x(t)$ so that $x(t) = \exp[A_2(t-t_1)]x(t_1) = \exp[A_2(t-t_1)]\exp[A_1(t_1-t_0)]x_0 = \exp[A_2(t-t_1)]\exp(A_1\Delta t_1)x_0$. Continuing similarly, for $t_{\sigma-1} \le t < t_\sigma$, $\dot{x}(t) = A_\sigma x(t)$ so that $x(t) = \exp[A_\sigma(t-t_{\sigma-1})]x(t_{\sigma-1}) = \exp[A_\sigma(t-t_{\sigma-1})]\exp[A_{\sigma-1}(t_{\sigma-1}-t_{\sigma-2})]\cdots\exp[(A_1(t_1-t_0)]x_0 = \exp[A_\sigma(t-t_{\sigma-1})]\exp(A_{\sigma-1}\Delta t_{\sigma-1})\cdots\exp(A_1\Delta t_1)x_0$. Thus, it follows that

$$R = \Phi(t_0+T, t_0) = \exp(A_\sigma \Delta t_\sigma)\exp(A_{\sigma-1}\Delta t_{\sigma-1})\cdots\exp(A_1\Delta t_1), \tag{3.6}$$

and

$$Q = \frac{1}{T}\log[\Phi(t_0+T,t_0)] = \frac{1}{T}\log[\exp(A_\sigma \Delta t_\sigma)\exp(A_{\sigma-1}\Delta t_{\sigma-1})\cdots\exp(A_1\Delta t_1)]. \tag{3.7}$$

Hence, the conclusion follows from the Floquet theorem. ■

Next, the above result is specialized to the case when the matrices A_1, A_2, $\cdots$, A_σ are pairwise commutative.

Corollary 3.1: The system (1.1) with

$$A_k A_l = A_l A_k, \quad k = 1,\cdots,\sigma,\ \ l = 1,\cdots,\sigma, \tag{3.8}$$

is exponentially stable if and only if the matrix

$$Q = \frac{1}{T}\sum_{k=1}^{\sigma} A_k \Delta t_k \tag{3.9}$$

is Hurwitz. If, in addition, $\Delta t_1 = \Delta t_2 = \cdots = \Delta t_\sigma = T/\sigma$, then the system (1.1) is exponentially stable if and only if the matrix

$$Q = \frac{1}{\sigma}\sum_{k=1}^{\sigma} A_k \tag{3.10}$$

is Hurwitz. ■

Proof: Since A_k and A_l are commutative for all $k = 1,\cdots,\sigma$ and $l = 1,\cdots,\sigma$, it follows from (3.1) that

$$R = \exp[\sum_{k=1}^{\sigma} A_k \Delta t_k]. \tag{3.11}$$

Thus, the matrix Q in (3.2) becomes

$$Q = \frac{1}{T} \sum_{k=1}^{\sigma} A_k \Delta t_k. \tag{3.12}$$

If, in addition, $\Delta t_k = T/\sigma$ for $k = 1, \cdots, \sigma$, then the matrix Q becomes

$$Q = \frac{1}{\sigma} \sum_{k=1}^{\sigma} A_k. \tag{3.13}$$

This completes the proof. ■

Given the subsystem matrices A_1, A_2, $\cdots$, A_σ and the activation durations Δt_1, Δt_2, $\cdots$, Δt_σ for these subsystems, the matrices Q and R defined above can be easily calculated. Hence, checking whether a given system is stable or not is relatively straightforward. However, the results of Theorem 3.1, except those of Corollary 3.1, are not very useful for designing periodically switched systems. This is because for a given set of matrices A_1, A_2, $\cdots$, A_σ, it is not easy to determine the activation durations Δt_1, Δt_2, $\cdots$, Δt_σ such that R is Schur or Q is Hurwitz. Below, two extreme cases, slow and fast switching, are considered. In each case, it is shown that exponential stability can be achieved under a very mild assumption.

Theorem 3.2: Consider the system (1.1) and assume that at least one of A_1, A_2, $\cdots$, A_σ is Hurwitz. Then, there exist a sufficiently large $T > 0$ and activation durations Δt_1, Δt_2, $\cdots$, Δt_σ such that the system (1.1) is exponentially stable. ■

Proof: For each $k = 1, \cdots, \sigma$, let the eigenvalues of the matrix A_k be λ_{k1}, λ_{k2}, $\cdots$, λ_{kn} and define

$$\alpha_k = \max_{i=1,\cdots,n} \mathrm{Re}\{\lambda_{ki}\}. \tag{3.14}$$

Then, for each $k = 1, \cdots, \sigma$, there exists a polynomial $\beta_k(\Delta t_k)$ such that

$$\| \exp(A_k \Delta t_k) \| \leq \beta_k(\Delta t_k) \exp(\alpha_k \Delta t_k). \tag{3.15}$$

Next, for a nonnegative integer r, consider

$$\begin{aligned} \|R\|^r &= \| \exp(A_\sigma \Delta t_\sigma) \exp(A_{\sigma-1} \Delta t_{\sigma-1}) \cdots \exp(A_1 \Delta t_1) \|^r \\ &\leq \| \exp(A_\sigma \Delta t_\sigma) \|^r \| \exp(A_{\sigma-1} \Delta t_{\sigma-1}) \|^r \cdots \| \exp(A_1 \Delta t_1) \|^r. \end{aligned} \tag{3.16}$$

Hence, it follows that

$$\|R^r\| \leq [\beta_1(\Delta t_1) \beta_2(\Delta t_2) \cdots \beta_\sigma(\Delta t_\sigma)]^r \exp[(\alpha_1 \Delta t_1 + \alpha_2 \Delta t_2 + \cdots + \alpha_\sigma \Delta t_\sigma) r]. \tag{3.17}$$

Since by assumption at least one of A_1, A_2, $\cdots$, A_σ is Hurwitz, at least one of α_1, α_2, $\cdots$, α_σ is negative. Hence, there exist a sufficiently large T and activation durations Δt_1, Δt_2, $\cdots$, Δt_σ such that $[\beta_1(\Delta t_1)\beta_2(\Delta t_2)\cdots\beta_\sigma(\Delta t_\sigma)]\exp(\alpha_1\Delta t_1+\alpha_2\Delta t_2+\cdots+\alpha_\sigma\Delta t_\sigma)<1$. This selection of T and Δt_1, Δt_2, $\cdots$, Δt_σ implies that

$$\lim_{r\to\infty} R^r = 0, \tag{3.18}$$

which means that R is Schur. Hence, by Theorem 3.1, the system is exponentially stable. ■

This theorem basically states that making the activation durations of the asymptotically stable subsystems sufficiently large compared to that of unstable ones results in an overall exponentially stable switched system. It should be pointed out that using Lyapunov theory, similar results have been obtained in literature for general switched systems [5], [6].

Theorem 3.3: Consider the system (1.1) and assume that $A_1\eta_1 + A_2\eta_2 + \cdots + A_\sigma\eta_\sigma$ is Hurwitz for $\eta_1 \geq 0$, $\eta_2 \geq 0$, $\cdots$, $\eta_\sigma \geq 0$ such that $\eta_1 + \eta_2 + \cdots + \eta_\sigma = 1$. Then, there exist a sufficiently small $T > 0$ and activation durations Δt_1, Δt_2, $\cdots$, Δt_σ such that the system (1.1) is exponentially stable. ■

Proof: Let $\Delta t_1 = \eta_1 T$, $\Delta t_2 = \eta_2 T$, $\cdots$, $\Delta t_\sigma = \eta_\sigma T$. Then, for a sufficiently small T, the matrix

$$Q = \frac{1}{T}\log[\exp(A_\sigma\Delta t_\sigma)\exp(A_{\sigma-1}\Delta t_{\sigma-1})\cdots\exp(A_1\Delta t_1)] \tag{3.19}$$

can be written as

$$\begin{aligned} Q &= \frac{1}{T}\log[I + A_1\Delta t_1 + A_2\Delta t_2 + \cdots + A_\sigma\Delta t_\sigma] + O(T^2) \\ &= (A_1\eta_1 + A_2\eta_2 + \cdots + A_\sigma\eta_\sigma) + O(T^2), \end{aligned} \tag{3.20}$$

where

$$\lim_{T\to 0}\frac{O(T^2)}{T} = 0. \tag{3.21}$$

Since by assumption $A_1\eta_1+A_2\eta_2+\cdots+A_\sigma\eta_\sigma$ is Hurwitz and since the eigenvalues of a matrix depends continuously on its elements, it follows that there exists a sufficiently small T such that Q is Hurwitz. Hence, by Theorem 3.1, the system is exponentially stable. ■

This theorem basically states that making the activation durations of every subsystems sufficiently small results in an overall exponentially stable switched system provided that the average of those subsystems is asymptotically stable. It should be pointed out that using averaging theory, similar results have been obtained in literature for general periodic systems [7], [8].

4. EXAMPLES

Two examples are given below to illustrate the salient futures of the developed results. Since the purpose is to show the usefulness of the developed theory, the chosen examples are fairly simple.

The first example considers the case where all subsystems of a switched system are asymptotically stable and demonstrates applications of Theorem 3.1 and Theorem 3.2. It also illustrates that for some switching rule, the switched system can be unstable even if all of its subsystems are asymptotically stable.

Example 4.1: Consider the system (1.1) with $\sigma = 2$ and

$$A_1 = \begin{bmatrix} -2 & -5 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -6 & 1 \\ -25 & 0 \end{bmatrix}. \tag{4.1}$$

Assume that $\Delta t_1 = \Delta t_2 = T/2$. The eigenvalues of A_1 are $-1 \pm j2$ and the eigenvalues of A_2 are $-3 \pm j4$. Note that although both A_1 and A_2 are Hurwitz, $A_1 + A_2$ is not Hurwitz. Thus, it follows from Theorem 3.2 that for sufficiently large T, the system is exponentially stable. The magnitudes of the eigenvalues of R as functions of T are plotted in Figure 4.1. From this figure, it follows that this system is unstable when $T = 1$ whereas it is exponentially stable when $T = 2$. A possible trajectory for each case is shown in Figure 4.2. ■

The second example, on the other hand, considers the case where all subsystems of a switched system are unstable and demonstrates applications of Theorem 3.1 and Theorem 3.3. It also illustrates that for some switching rule, the switched system can be stable even if all of its subsystems are unstable.

Example 4.2: Consider the system (1.1) with $\sigma = 2$ and

$$A_1 = \begin{bmatrix} 2 & -5 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3.5 & 2 \\ 1 & 0 \end{bmatrix}. \tag{4.2}$$

Assume that $\Delta t_1 = \Delta t_2 = T/2$. The eigenvalues of A_1 are $1 \pm j2$ and the eigenvalues of A_2 are 0.5, -4. Note that although neither A_1 nor A_2 is Hurwitz, $A_1 + A_2$ is Hurwitz. Thus, it follows from Theorem 3.3 that for sufficiently small T, the system is exponentially stable. The magnitudes of the eigenvalues of R as functions of T are plotted in Figure 4.3. From this figure, it follows that this system is exponentially stable when $T = 1$ whereas it is unstable when $T = 2$. A possible trajectory for each case is shown in Figure 4.4. ■

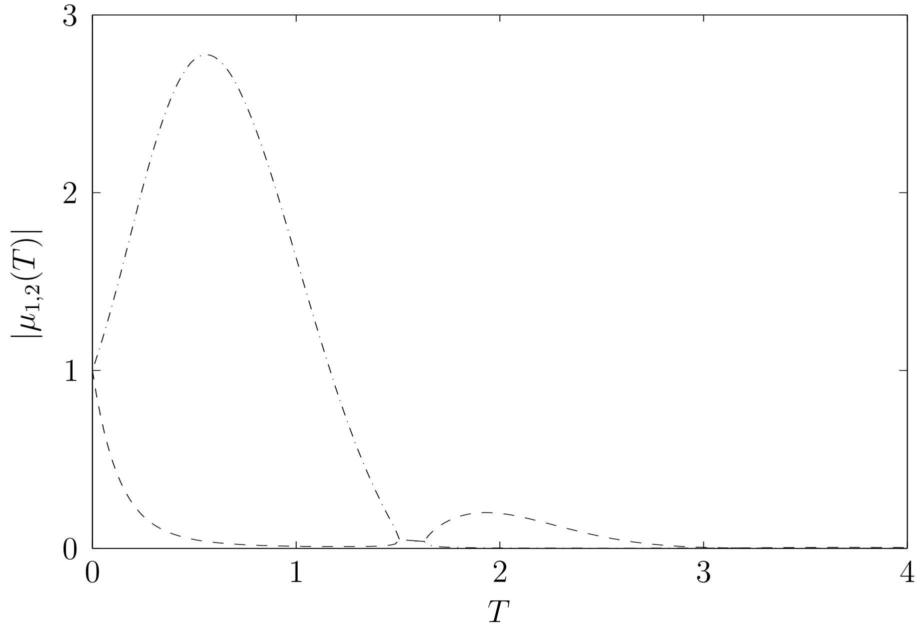

Figure 4.1: Magnitudes of the eigenvalues of R.

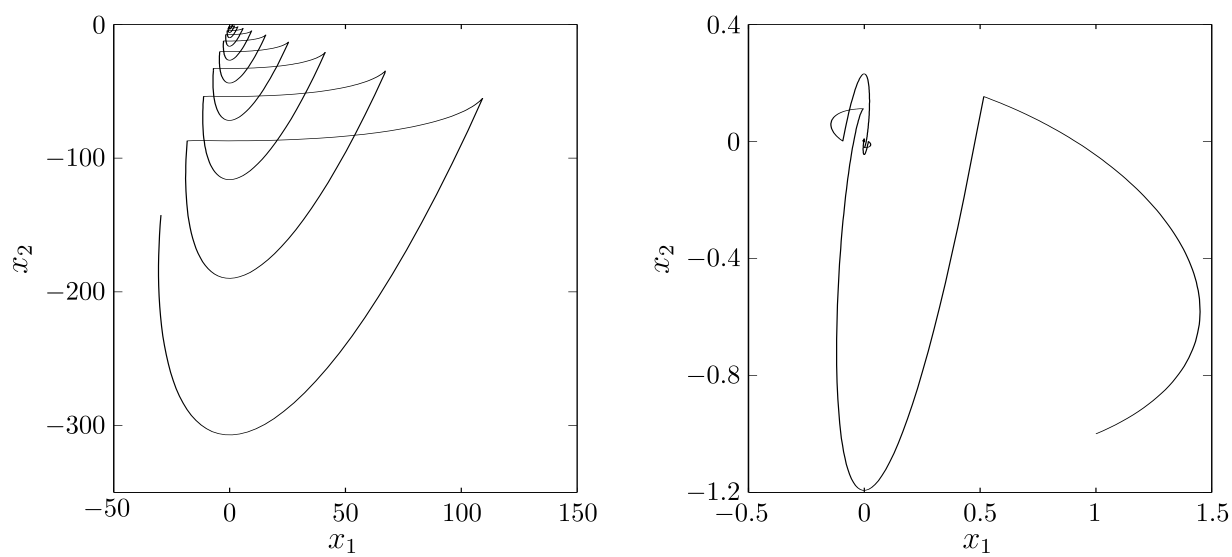

Figure 4.2: Phase portraits for $T = 1$ left and for $T = 2$ right.

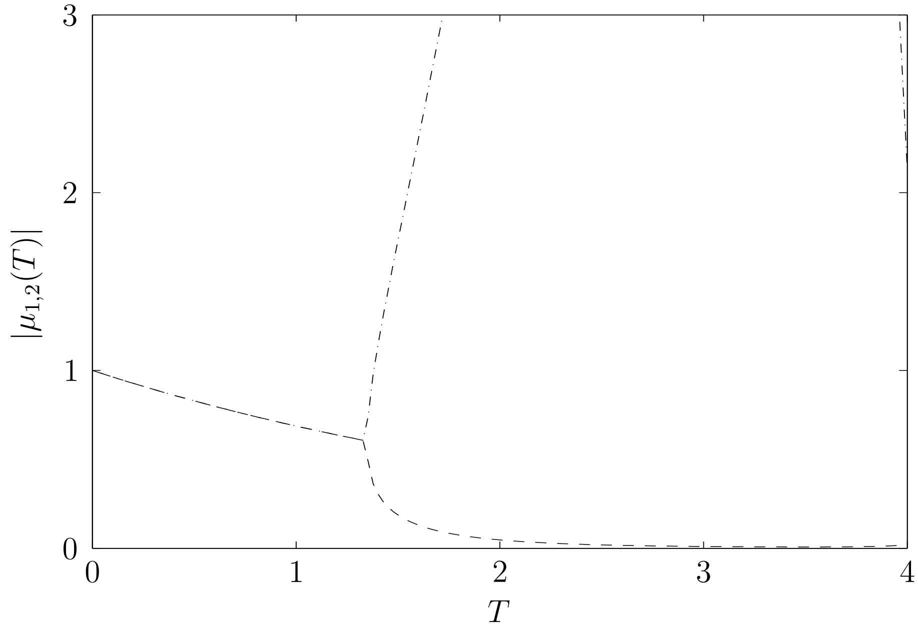

Figure 4.3: Magnitudes of the eigenvalues of R.

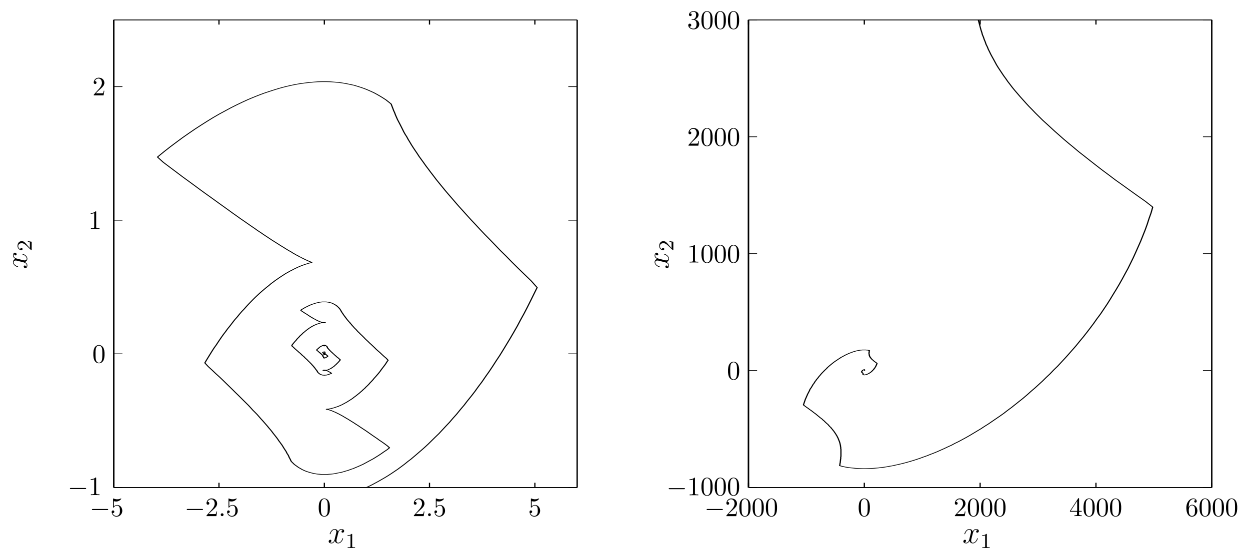

Figure 4.4: Phase portraits for $T = 1$ left and for $T = 2$ right.

5. CONCLUSIONS

In this paper, stability of periodically switched linear systems is considered. Using the Floquet theory, necessary and sufficient conditions are derived for exponential stability. It is shown that there exists a slow switching rule that achieves exponential stability if at least one of the subsystems is asymptotically stable. It is also shown that there exists a fast switching rule that achieves exponential stability if the average of the subsystems is stable.

REFERENCES

[1] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Contr. Syst. Mag.,* vol. 19, no. 5, pp. 59–70, 1999.

[2] A. S. Morse, ed., *Control Using Logic-Based Switching.* London: Springer-Verlag, 1997.

[3] J. A. Richards, *Analysis of Periodically Time-Varying Systems.* NY: Springer-Verlag, 1983.

[4] W. J. Rugh, *Linear System Theory.* NJ: Prentice Hall, 1996.

[5] P. Peleties and R. A. DeCarlo, "Asymptotic stability of m-switched systems using Lyapunov-like functions," *Proc. Amer. Contr. Conf.,* Boston, MA, pp. 1679–1684, 1991.

[6] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Contr.,* vol. 43, no. 4, pp. 475–482, 1998.

[7] R. Bellman, J. Bentsman, and S. M. Meerkov, "Stability of fast periodic systems," *IEEE Trans. Automat. Contr.,* vol. 30, no. 3, pp. 289–291, 1985.

[8] H. K. Khalil, *Nonlinear Systems Analysis.* NJ: Prentice-Hall, 1996.