Parametric Identification of Chaotic Systems *

C.-M. YUAN
Chung-Shan Institute of Science and Technology
Taoyuan, Taiwan R. O. C.

and

B. F. FEENY
Department of Mechanical Engineering
Michigan State University
2555 Engineering Building
East Lansing, MI 48824 U.S.A.

Abstract. Parameters are identified in chaotic systems. Periodic orbits are first extracted from a chaotic set. The harmonic-balance method is applied to these periodic orbits, resulting in a linear equation in the unknown parameters, which can then be solved in the least-squares sense. The idea is applied numerically to forced and autonomous systems. The effects of noise and errors in the periodic orbit extraction are outlined. The benefit of extracting several periodic orbits from the chaotic set is revealed.

Key Words: chaos, nonlinear parametric identification, periodic-orbit extraction, harmonic balance

1. INTRODUCTION

Chaotic systems have been of interest for many years. The range of focus on studies of chaos has included behavior, control, modeling, and applications. Our premise in this work is that chaotic responses contain valuable information which can be used in parametric identification.

In parametric system identification, enough is known *a priori* to write the form of the differential equations of motion, with the coefficients multiplying linear and nonlinear functions in these equations to be identified. Among the many methods for nonlinear parametric identification are those of Nayfeh (1985), in which resonances are exploited, Stry and Mook (1992), which is applied to the time series, and Yasuda *et al.* (1988a, 1988b), in which the harmonic-balance method is used in an inverse way to estimate parameters.

One of the applications of chaotic systems is in system modeling. To this end, much of the activity is geared toward dimensionality studies, in which bounds on the number of active state variables are established. The determination of the size of a system might be a first step in developing *a priori* knowledge that may later be used in parametric system identification. In addition, nonlinear prediction can be performed using modern methods of analyzing chaotic data (Abarbanel *et al.*, 1993).

The fundamental property of deterministic chaos that we exploit is that a chaotic set is the closure of infinitely many periodic orbits. Poincaré (1854-1912), the discoverer of what we now call chaos in the three-body problem ("The complexity of this figure will be striking, and I shall not even try to draw it"), realized the existence of periodic orbits within this complexity, and their potential usefulness: "these periodic solutions are so valuable for us because they are, so to say, the only breach by which we may attempt to enter an area

* *Journal of Vibration and Control,* **4:** 405-426, 1998

heretofore deemed inaccessible.” (Quotes obtained through Tufillaro *et al.*, 1992).

Indeed the periodic orbits can be extracted from chaotic data from either discrete or continuous time systems (Auerbach *et al.*, 1987; Lathrop and Kostelich, 1989; Tufillaro *et al.*, 1992). They have been used in system identification (Hammel and Heagy, 1992; Kesaraju and Noah, 1994; Feeny and Yuan, 1994; Van de Wouw *et al.*, 1995).

This paper is a follow-up to work presented previously (Feeny and Yuan, 1994). In our work, we extract unstable periodic orbits from a chaotic set. These periodic orbits, although unstable, represent solutions or responses in the system. We then use the harmonic-balance method (as in Yasuda *et al.*, 1988a; Yasuda *et al.*, 1988b) on these periodic responses. We thus extend Yasuda's ideas to the extracted periodic orbits.

In the next section, we describe the algorithm which ties periodic-orbit extraction with Yasuda's method of parameter estimation. We also outline the adjustments needed for an autonomous system. In section 3, we apply the method to numerical examples. Section 4 contains error analyses. We close with conclusions.

2. PARAMETER IDENTIFICATION SCHEME

2.1. Extraction of the Periodic Orbits

The unstable periodic orbits can be extracted using the recurrence property of the chaotic attractor. We scan the data set to locate the recurrences within a “small” spatial distance ϵ, such that

$$\|x_{i+K} - x_i\| \leq \epsilon.$$

Satisfying this inequality implies that the data between x_i and x_{i+K} are “close” to a periodic orbit with a period of K samples.

In a periodically forced system, all periodic orbits have a period that is an integer multiple of the forcing period, such that $K = kn_0$, where n_0 is the number of samples per forcing period, and k is an integer. However, in an autonomous system, there is no forcing period. A chaotic set from an autonomous system generally consists of periodic orbits with incommensurate periods. The periods of the periodic orbits can be obtained with the help of a recurrence plot. Such a plot is constructed by varying the delay K from zero to some practical maximum, and counting the number of recurrent points found from the data scan for each value of K. The recurrence periods will be clustered at certain values, indicating the periodicity of the periodic orbits and hence the number of points K in each period. Each resulting value of K is used for extracting the periodic orbit of period K.

This procedure generally works well for extracting periodic orbits. However, if the positive Lyapunov exponent associated with an unstable periodic orbit is “large,” then in one period a nearby orbit will likely depart from the unstable periodic orbit, and will go unidentified. In such case, the spatial criterion ϵ can be relaxed to include more points in the neighborhood. For finding low-order periodic orbits, say less than ten, it is typically adequate to set ϵ to be 0.5% of the span of the chaotic set (Auerbach *et al.*, 1987; Kesaraju and Noah, 1994).

The searching process may reveal several distinct unstable periodic orbits with the same period. Nonetheless, all extracted periodic orbits can be used in the parametric identification scheme.

2.2. Identification in Periodically Forced Systems

We illustrate the method introduced by Yasuda *et al.* (1988a) through a single-degree-of-freedom, externally excited system. The mathematical model is chosen as

$$m\ddot{x} + \sum_{i=1}^{p} \beta_i f_i(x, \dot{x}) = E(t). \tag{1}$$

where the external exciting function, $E(t)$, is considered to be periodic with single frequency ω, such that $E(t) = a\cos(\omega t)$, and p is the number of functions in the differential equation.

The motion variables and the nonlinear functions f_i, when evaluated at the periodic orbits from the chaotic set, are periodic and can be approximated in a truncated Fourier series, such as

$$x_k \approx \frac{a_0}{2} + \sum_{j=1}^{n} (a_j \cos\frac{j\omega t}{k} + b_j \sin\frac{j\omega t}{k})$$

$$\dot{x}_k \approx \sum_{j=1}^{n} \frac{j\omega}{k} (b_j \cos\frac{j\omega t}{k} - a_j \sin\frac{j\omega t}{k})$$

$$\ddot{x}_k \approx -\sum_{j=1}^{n} \frac{j^2\omega^2}{k^2} (a_j \cos\frac{j\omega t}{k} + b_j \sin\frac{j\omega t}{k})$$

$$f_i(x, \dot{x})_k \approx \frac{c_{i0}}{2} + \sum_{j=1}^{n} (c_{ij} \cos\frac{j\omega t}{k} + d_{ij} \sin\frac{j\omega t}{k})$$

with the Fourier coefficients calculated as

$$a_j = \frac{2}{T}\int_{\phi}^{\phi+T} x_k(t)\cos\frac{j\omega t}{k}dt$$

$$b_j = \frac{2}{T}\int_{\phi}^{\phi+T} x_k(t)\sin\frac{j\omega t}{k}dt$$

$$c_{ij} = \frac{2}{T}\int_{\phi}^{\phi+T} f_i(x, \dot{x})_k(t)\cos\frac{j\omega t}{k}dt$$

$$d_{ij} = \frac{2}{T}\int_{\phi}^{\phi+T} f_i(x, \dot{x})_k(t)\sin\frac{j\omega t}{k}dt.$$

The subscript k indicates that the data is period-k. T is the period, and $\omega = 2\pi/T$ is the fundamental frequency.

These Fourier coefficients are to be implemented into a harmonic balance scheme. It is thus necessary that the phase of the extracted orbit is consistent with the phase of the forcing term. Ignoring the phase angle will cause an inconsistency which will cause incorrect identification results.

Substituting these Fourier series into the model equation (1), and balancing the Fourier coefficients of any set of harmonics, a set of *linear* algebraic equations in the system parameters can be constructed. This application of the harmonic-balance method contrasts its usual usage for response analysis, in which the ordinary differential equation is known, and the goal is to solve a set of *nonlinear* equations in Fourier coefficients. For systems forced with a single harmonic, and for autonomous systems, the method of harmonic balance requires nonlinearity so that several harmonics can be balanced. A linear single-degree-of-freedom system with a harmonic response yields only two balance equations, and thus only two parameters could be estimated.

In this paper, we typically use multiples of the primary harmonic. Thus, for the example of $k = 1$, the balance equations are

$$\begin{bmatrix} 0 & c_{00} & \cdots & c_{p0} \\ -\omega^2 a_1 & c_{11} & \cdots & c_{p1} \\ -\omega^2 b_1 & d_{11} & \cdots & d_{p1} \\ \vdots & & & \\ -(n\omega)^2 a_1 & c_{1n} & \cdots & c_{pn} \\ -(n\omega)^2 b_1 & d_{1n} & \cdots & d_{pn} \end{bmatrix} \begin{bmatrix} m \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or

$$A\alpha = q, \tag{2}$$

where α is the parameter vector of the system model, A is a $(2n + 1) \times (p + 1)$ matrix, q is a $(2n + 1)$ vector containing the Fourier coefficients of the external forcing function, and n is the number of terms retained in the Fourier series representation. Each column of A contains the Fourier coefficients of the corresponding term in the system model. The vector q contains a non-zero element a in our periodic-excitation example.

If any parameters are known, they can be incorporated into the known quantities in q. It is wise to nondimensionalize the system to reduce the number of parameters. For example, simply by dividing through by m, we would eliminate one parameter, and end up with a coefficient of '1' in front of the $\ddot{x}$ term. In such case, the first column of A can be subtracted from q. For demonstrational purposes, we have retained m in this example as an unknown parameter.

If $2n = p$ and the matrix A is non-singular, the parameter vector α can be determined uniquely. In practice, it is statistically better if the algebraic equation (2) is overdetermined, so that $2n > p$. Consequently the exact solution will not generally exist, but a best solution can be obtained by a least-squares fit. We seek a solution that can minimize the average error in all of the equations. The error function is most conveniently chosen as the sum of squares, or defined as (Atkinson, 1989)

$$e = ||A\alpha - q||.$$

The solution that minimizes this equation is called the least-squares solution of the linear system. The minimization of the error function is performed by setting the partial derivatives of the squared error function with respect to the parameters equal to zero, i.e., $\partial e^2/\partial\alpha = 0$, which leads to the *normal equations*

$$A^T(A\alpha - q) = 0.$$

The least-squares solution of the parameters vector α is

$$\alpha = (A^T A)^{-1} A^T q.$$

Since the operation of a matrix inversion is time consuming and of low accuracy, the normal equation is often not recommended in the numerical implementation. The most general least-squares solution based on the singular-value decomposition method is

$$\alpha = V\Sigma^{\dagger} U^T q = A^{\dagger} q,$$

where through the singular-value decomposition, $A = U\Sigma V^T$, and $A^{\dagger}$ is its pseudo-inverse; U and V are the orthogonal matrices with each column consisting of the left and right singular vectors of matrix A respectively; and $\Sigma^{\dagger}$ is the pseudo-inverse of Σ. $\Sigma^{\dagger}$ has the non-negative diagonal elements being the inverse of that of the corresponding terms in Σ. (See Atkinson, 1989, and Strang, 1976, for a geometric discussion.)

A question arises as to how many terms should be retained in the Fourier series representation of a periodic solution. Mickens (1988) has shown that the upper bounds of the absolute magnitudes of the harmonic coefficients decrease exponentially as n increases, such that they become ineffective in the least-squares estimation procedures. The rule of thumb for retaining the number of terms in the Fourier series is $3 \leq n \leq 5$, where n is the number of harmonics of the primary (driving) frequency. This limits the number of unknown parameters in the model that can be estimated using a single periodic orbit. However, we can use several periodic orbits to form several sets of algebraic equations, thereby augmenting the matrix A to increase the redundancy of algebraic equations for the least-squares estimation. This treatment can improve the estimation results. Moreover, when the parameter set is small, each set of algebraic equations formed by individual periodic orbit can be used separately to obtain statistical information such as mean values and standard deviations. This availability of several extracted periodic orbits from a chaotic set increases the applicability beyond that of a simple periodic response.

These ideas can be applied to parametrically excited systems, which have also been shown to exhibit chaotic behavior in a large range of parameters (Cusumano and Sharkady, 1995; Malasoma *et al.*, 1994).

The mathematical model for a single-degree-of-freedom parametrically excited system is chosen as

$$\ddot{x} + \sum_{i=1}^{r} g_i(t) \sum_{j=1}^{p_i} \beta_{ij} f_{ij}(x, \dot{x}) = 0, \tag{3}$$

where each $g_i(t)$ is periodic. The $\ddot{x}$ term is taken as a known quantity by the fact that the approximate periodic solution of the original system is known, and moved to the right-hand side of the equation.

Using the extracted periodic orbits, the evaluated nonlinear functions in the model are periodic. The excitation functions in time and the nonlinear functions in x and $\dot{x}$ are combined together when they are to be expressed in Fourier series, such that

$$\tilde{g}_{ij}(x, \dot{x}, t) = g_i(t) f_{ij}(x, \dot{x}) \approx \frac{c_{ij0}}{2} + \sum_{k=1}^{n} (c_{ijk} \cos \frac{j\omega t}{k} + d_{ijk} \sin \frac{j\omega t}{k}),$$

whence the Fourier coefficients can be calculated. Substituting the Fourier series into the model equation (3), and balancing the Fourier coefficients of each harmonic, a set of algebraic equations such as equation (2) can be constructed, and the parameters can be estimated by a least-squares fit.

2.3. Autonomous Systems

An autonomous system of dimension three or more can exhibit chaotic behavior (Guckenheimer and Holmes, 1983). The famous example is the Lorenz equation, given by

$$\begin{aligned} \dot{y}_1 &= \sigma(y_2 - y_1) \\ \dot{y}_2 &= \rho y_1 - y_2 - y_1 y_3 \\ \dot{y}_3 &= \beta y_3 + y_1 y_2. \end{aligned} \tag{4}$$

There exists a chaotic attractor in some region of the parameter space. To extract the unstable periodic orbits, the incommensurate periods have to be determined by constructing a recurrence plot, as stated in section 2.1.

The mathematical model is chosen as

$$\dot{y}_i = \sum_{j=1}^{p_i} \beta_{ij} h_{ij}(y), \quad i = 1, \ldots, n, \quad n \geq 3, \tag{5}$$

where $y = [y_1, \ldots, y_n]^T$, $h_{ij}(y)$ are the nonlinear functions of the state variable y, and p_i are the number of terms in each equation of the model.

In an experiment, each state variable y_i must be measured. Using the periodic orbits extracted from the chaotic attractor, each term in the model is periodic and can be expressed in a Fourier series with the fundamental frequency as the one obtained from the recurrence plot. The Fourier coefficients are calculated as before, except the phase angle can be ignored since there is no forcing function involved.

3. NUMERICAL RESULTS

In this section, numerical studies on the forced Duffing oscillator and the Lorenz equation are used to demonstrate the applicability of this approach. Numerical integration of the governing equations were carried out using a 5th-order Runge-Kutta method on a Sun workstation. Typically, 50000 chaotic data points were generated for a time step interval of one one-hundredth of the forcing period, or with a time step size of 0.005 in autonomous systems.

3.1. The Forced Duffing Oscillator

The forced Duffing oscillator is given as

$$m\ddot{x} + c\dot{x} + \beta x + \gamma x^3 = a \cos \omega t. \tag{6}$$

With $\beta = 0$, equation (6) is a model for a circuit with a nonlinear inductor (Ueda, 1992), and with $\beta < 0$ and $\gamma > 0$ it is a model for the postbuckling vibrations of an elastic column under compressive loads (Moon, 1992).

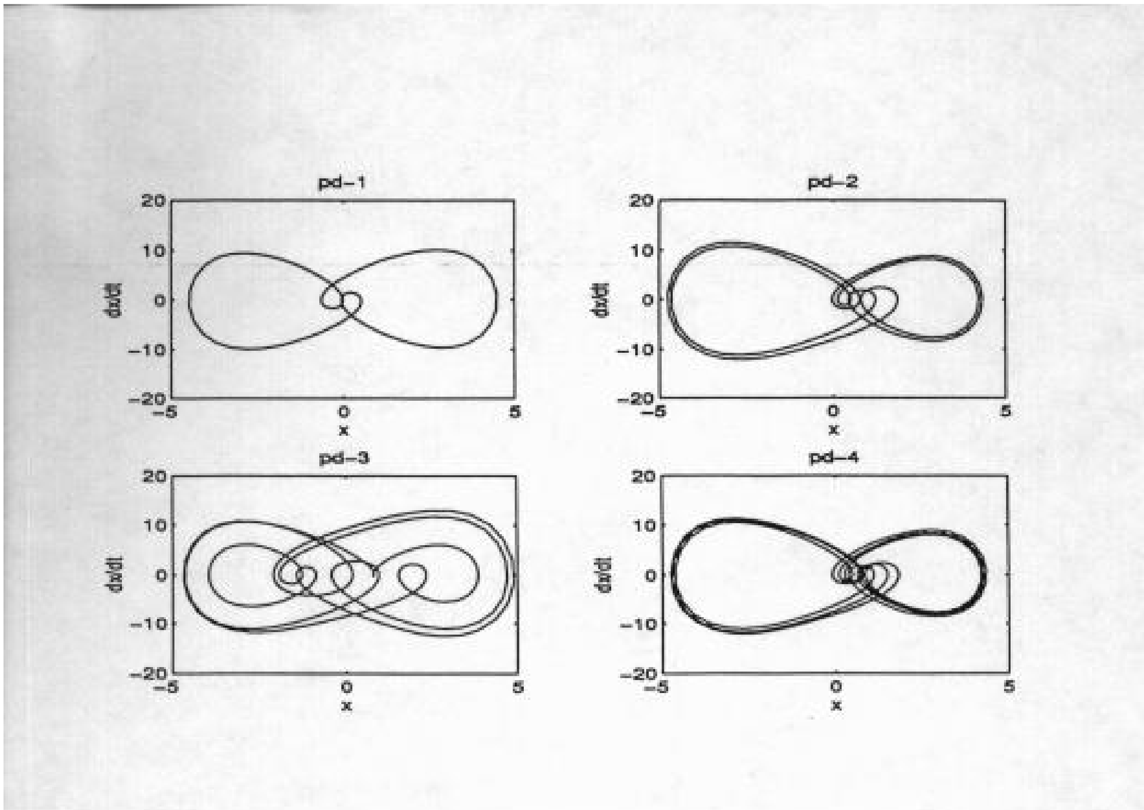

Figure 1. Some extracted periodic orbits of the Duffing oscillator.

This equation admits chaotic motions for a large range of parameters. We choose the parameter values as $m = 1, c = 0.2, \beta = \gamma = 1, a = 27$, and $\omega = 1.33$ (Tufillaro *et al.*, 1992). These parameters are to be estimated by the present method.

An attracting set was generated numerically. This is the attracting set from which the unstable periodic orbits are to be extracted. The Lyapunov exponents, indicating the average exponential rates of divergence or convergence of nearby orbits in phase space, were calculated using the computer code by Wolf *et al.* (1985). They converged to $\lambda_1 = 0.18$, and $\lambda_3 = -0.468$, indicating that the system is sensitive to initial conditions. The exponent $\lambda_2 = 0$ resulted automatically in our computation.

Several unstable periodic orbits were extracted using a spatial distance ϵ chosen as 0.5% of the span of the data set. Some of the extracted periodic orbits are shown in Figure 1.

For the parametric identification, we chose a model in polynomial form, written as

$$m\ddot{x} + \alpha\dot{x} + \sum_{i=1}^{p} \beta_i x^i = a \cos \omega t, \tag{7}$$

where m, α, and β are the parameters to be identified.

Table 1. Identification results using individual periodic orbits for Duffing's equation.

data	m	α	β_1	β_2	β_3	β_4	β_5
actual	1.0000	.2000	1.0000	0.0000	1.0000	0.0000	0.0000
pd-3	0.9999	.1997	0.9915	-.0023	1.0009	.0002	.0000
pd-4	1.0008	.2002	1.0498	-.0050	0.9803	.0007	.0010
pd-5	0.9997	.1998	0.9659	-.0020	1.0124	.0001	-.0006
pd-6	1.0001	.1999	1.0015	.0016	1.0009	-.0002	-.0061
Average	1.0001	.1999	1.0022	-.0019	0.9968	.0002	-.0014
st. dev.	0.0005	.0002	0.0351	.0027	0.0134	.0004	.0032

Table 2. Identification results using four periodic orbits for Duffing's equation.

p	m	α	β_1	β_2	β_3	β_4	β_5	β_6	β_7
actual	1.0000	.2000	1.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
4	1.000	0.200	1.005	-0.001	1.000	0.000			
5	1.000	0.200	1.015	-0.001	0.996	0.000	0.000		
6	1.000	0.200	1.010	0.004	0.998	-0.001	0.000	0.000	
7	1.000	0.200	1.005	0.004	1.000	-0.001	0.000	0.000	-0.000
Avg.	1.000	0.200	1.009	.0015	0.999	-.001	0.000	0.000	0.000
std.dv	0.000	0.000	0.005	0.003	0.002	0.001	0.000	0.000	0.000

3.1.1. Identification Results

We first apply four sets of the periodic-orbit data separately to the model of equation (7), with five polynomial terms retained. The identification results are shown Table 1.

Also, we apply four sets of periodic-orbit data together to increase the redundancy of the least-squares fit with different number of terms included in the model. The identification results are shown in Table 2.

The results are consistent with each other among the various sets of periodic-orbit data. The non-zero parameter values are recovered within 1% of their nominal values, and the zero-valued parameters are close to zero, even when the mathematical model contains many unnecessarily high-degree nonlinear terms. The standard deviations are less than 1% of the non-zero parameter values, or close to the average values of the zero-valued parameters, indicating the consistency of the results.

Combining individual sets of algebraic equations increases the redundancy in the least-

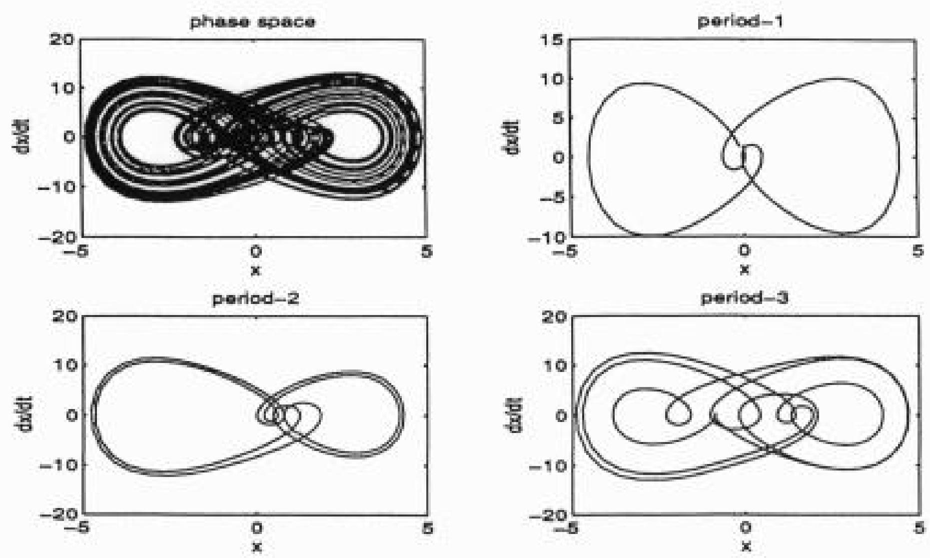

Figure 2. The simulated chaotic attractor and some of the extracted periodic orbits of the identified model.

squares fit, and improves the accuracy of the identification results when the model includes many parameters.

The identified results suggest that the model can be refined by removing the higher-degree nonlinear terms whose parameter values are negligible. The reduction of the unnecessary terms in the model tends to yield higher accuracy in the identification results.

3.1.2. Model Verification

From the numerical results, the model (7) can be easily verified. However, we use the identified model with the average values in Table 2 to generate a set of data, and extract the unstable periodic orbits, for comparison with the original ones. The extracted unstable periodic orbits from the identified model are shown in Figure 2.

It has been suggested that bifurcation diagrams are very sensitive to modeling errors (Cusumano and Sharkady, 1995) and may be used for model validation (Aguirre and Billings, 1994). The bifurcation diagrams shown in Figure 3 were calculated using the original equation and the identified one, by slowly increasing the forcing amplitude and sampling the steady-state response at the same Poincaré section.

For a quantitative comparison, the Lyapunov exponents of the identified model calculated by the computer code of Wolf *et al.* (1985) are convergent to $\lambda_1 = 0.20$, and $\lambda_3 = -0.49$ which are close to the original values of $\lambda_1 = 0.18$ and $\lambda_3 = -0.468$, with deviations of 11.1% and 4.7% respectively.

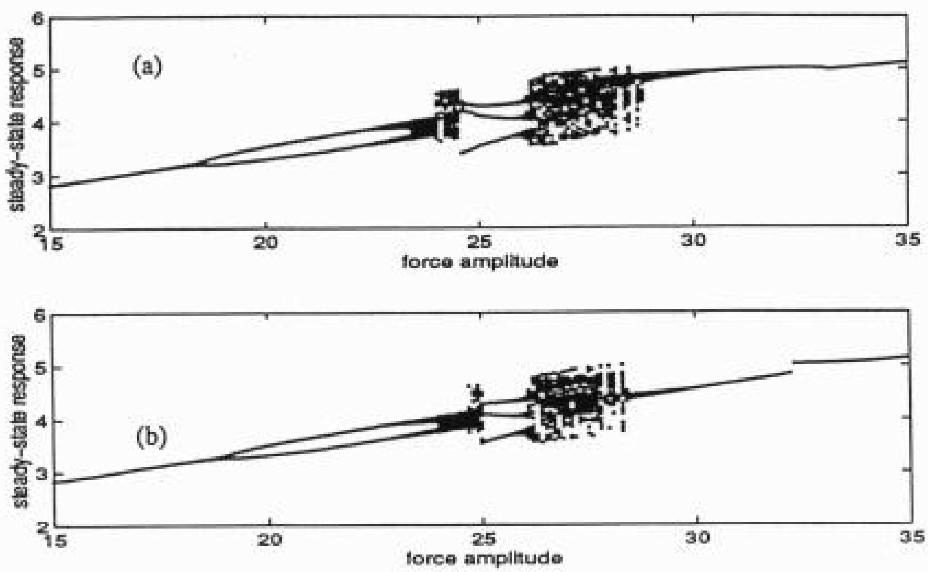

Figure 3. Bifurcation diagrams of the Duffing's equation using (a) the original equation, and (b) the identified model with the average values in Table 2.

3.1.3. Effect of Noise

Numerically generated data are considered to be essentially noise free. The excellent identification results in the example above may deteriorate if noise is present in the periodic data. To assess the influence of noise on the identification results, a set of uniformly-distributed random noise is added to each periodic orbit for use in the identification algorithm to test sensitivity of A, q and $A\alpha = q$. If the noise is added to the chaotic set before the extraction of the periodic orbits, the spatial criterion ϵ may need an adjustment. Large levels of noise may adversely effect the periodic orbit extraction (Lathrop and Kostelich, 1989).

The noise level is set by the ratio of its maximum amplitude to that of the employed periodic-orbit data. Figure 4(a) shows a period-3 noise-free periodic orbit and Figure 4(b) shows its 2% noise-contaminated counterpart. We examine the noise effect by using the model (7) with varying nonlinear terms and varying noise levels. Four sets of noisy periodic-orbit data are used together in the identification algorithm. The identification results are shown in Table 3.

Comparing with the previous results in Table 2 for the same model, we draw the following conclusions.

1. Within 5% noise level, the noise effect is not significant for a model with three or four terms in the polynomial. As the nonlinear terms increase beyond five, the noise effect increases. The errors are within 2.3% of the non-zero parameters. The zero-valued parameters have larger deviations than in previous case.

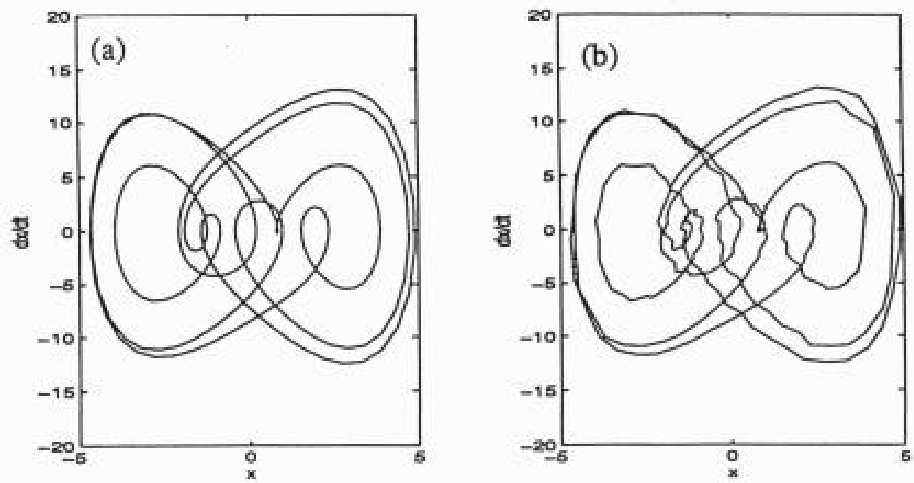

Figure 4. (a) A noise-free periodic orbit and (b) the noise-contaminated counterpart.

Table 3. Identification results for Duffing's equation with noisy data.

noise	p	m	α	β_1	β_2	β_3	β_4	β_5
	actual	1.0000	.2000	1.0000	0.0000	1.0000	0.0000	0.0000
1%	3	1.000	0.200	0.999	-0.006	1.000		
1%	5	1.000	0.200	1.058	-0.016	0.983	0.001	0.001
3%	3	1.000	0.200	0.995	-0.020	1.002		
3%	5	1.001	0.201	1.136	-0.044	0.958	0.002	0.002
5%	3	1.000	0.200	0.997	-0.032	1.002		
5%	5	1.001	0.201	1.202	-0.073	0.931	0.003	0.003

2. For five or more nonlinear terms in the model, the noise effect increases rapidly, resulting in less accurate identification results.

3. For a small amount of noise, the effect of noise is not catastrophic to our method.

The effect of noise on the parameter estimates will be analyzed in Section 4.

With the model identified using the noisy periodic data, we verify the model by comparing the Lyapunov exponents, the structure of the unstable orbits, and the bifurcation diagrams as before. Using a model with the parameter values as in the last second row of Table 3, the Lyapunov exponents are convergent to $\lambda_1 = 0.21$, and $\lambda_3 = -0.5$, which are close to the original values with deviations of 16.67% and 6.84% respectively.

The simulated chaotic attractor, its extracted periodic orbits, and the bifurcation diagram from the identified model qualitatively resemble the original ones, and are not displayed here (See Yuan, 1995).

3.1.4. Remarks

The primary harmonics were used in the identification. We could have used harmonics of the fundamental frequency. This would have increased the number of equations used in the least-squares fit. This increased redundancy should generally improve the results.

Studies on the effects of modeling errors are not presented here. A brief examination of this issue is presented through an example in Yuan (1995). When a power series model is implemented, one of the concerns is whether a power series converges to the actual underlying function in the range of the data. If a power series is divergent, there may be a best choice on the number of power series terms to be included. There also may be a limit on the quantitative accuracy of the model, although qualitative results may be obtained. Yasuda and Kawamura (1989) proposed the usage of linear interpolating functions, an approach which introduces many unknown parameters.

Other examples, such as a parametrically excited system, are included in Yuan (1995).

3.2. An Autonomous System: the Lorenz Equations

We take the Lorenz equation as an example of an autonomous chaotic system, written as equation (4) with the parameter values as $\sigma = 16, \rho = 45.92$, and $\beta = -4$. Numerical data are generated using a 5th-order Runge-Kutta method for 10000 data points with time interval being 0.025 sec.

To extract the unstable periodic orbits, we first construct a recurrence plot to determine the period length of the periodic orbits, as stated in detail in section 2.1. The recurrence plot is shown in Figure 5, in which recurrent points are clustered at certain delay values. These values indicate the periods of the periodic orbits. Using the values, the corresponding periodic orbits can be located within the Lorenz attractor. Some of the extracted periodic orbits are shown in Figure 6.

A mathematical model is chosen such that the linear terms and the quadratic nonlinear terms are included as

$$\dot{x}_m = \sum_{i=1}^{3}(a_{mi}x_i + \sum_{j=1}^{3} b_{mij}x_ix_j), \quad m = 1, 2, 3, \tag{8}$$

where a_{mi} and b_{mij} are the parameters to be determined.

Using the periodic orbits, each term in the model is expressed in a Fourier series with the fundamental period obtained from the recurrence plot. The Fourier coefficients are then calculated. By balancing the Fourier coefficients of each harmonic in each equation represented by equation (8), and treating the $\dot{x}_m$ terms as known quantities, a set of algebraic equations in the system parameters is constructed for the least-squares estimation.

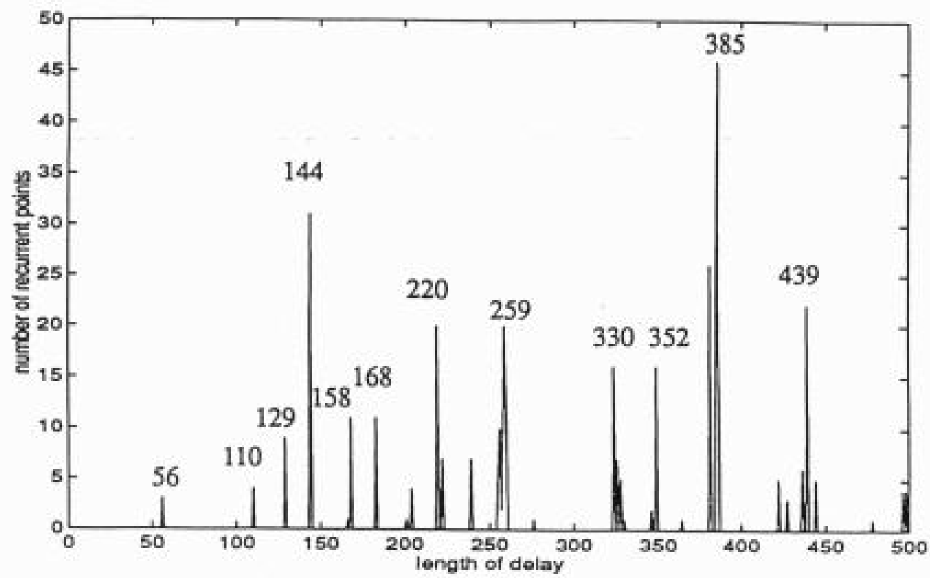

Figure 5. Recurrence plot of the Lorenz system (period lengths are indicated in the numbers of time steps used in the numerical integration).

Table 4. Identification results for the Lorenz equation.

	x_1	x_2	x_3	x_1x_1	x_1x_2	x_1x_3	x_2x_2	x_2x_3	x_3x_3
$\dot{x}_1$	-15.951	15.971	-.098	-.030	.019	-.003	.000	.003	.003
	(-16.0)	(16.0)							
$\dot{x}_2$	45.748	-.927	.230	.039	-.028	-.995	.003	-.004	-.006
	(45.92)	(-1.0)				(-1.0)			
$\dot{x}_3$	-.089	.041	-3.877	.031	.980	.003	.000	-0.003	-.004
			(-4.0)		(1.0)				

We use two periodic orbits with period lengths of 110 time steps and 144 time steps respectively, as shown in Figure 6, in the identification algorithm. The estimated parameter values are shown in Table 4.

The actual nonzero parameters from the original system are indicated in parentheses.

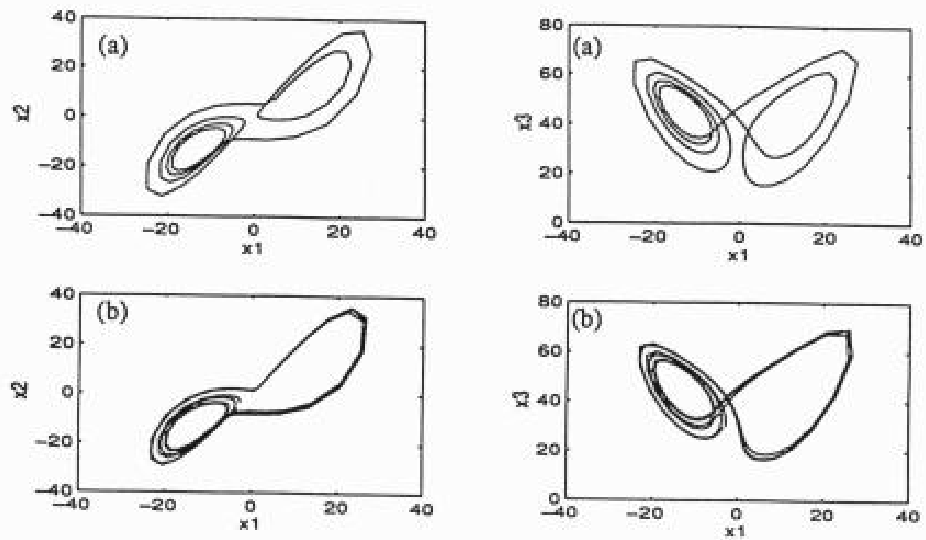

Figure 6. Two extracted periodic orbits of the Lorenz system: (a) period length of 110 time steps, (b) period length of 144 time steps.

Some of the zero-valued parameters are not close to zero, such as the third term in the second equation.

Suppose, for example, we knew *a priori* that there should be no squared terms in the model (8). This refinement improves the accuracy of the identification results significantly. Not only are the non-zero parameters closer to the real values, but also the zero-valued parameters are close to zero, as shown in Table 5.

3.2.1. Effect of Noise

To assess the influence of noise on the identification results, we add a set of uniformly-distributed random noise to the periodic orbits as before. With 1% noise added to the extracted periodic orbits, the identification results of the model without square-terms are not significantly affected, although some of the zero-value terms have non-zero values, as shown in Table 6. With higher-level noise added to the periodic orbits, the identification results deteriorate rapidly.

4. ERRORS IN PARAMETER ESTIMATES

Errors are inevitable in any parametric identification method, arising from incorrect modeling, data acquisition and data manipulations. Modeling is not only a main source of error, but also a critical factor for the success of an identification method.

In this section, we focus on the quality of the data, and its effect on the formulation

Table 5. Identification results for the Lorenz equation. *A priori* modeling is improved by omitting the squared terms.

	x_1	x_2	x_3	x_1x_2	x_1x_3	x_2x_3
$\dot{x}_1$	-16.011	15.953	-0.006	0.001	0.0002	0.0011
	(-16.0)	(16.0)				
$\dot{x}_2$	45.900	-1.0228	-0.0127	0.0020	-0.9996	0.0021
	(45.92)	(-1.0)			(-1.0)	
$\dot{x}_3$	0.0041	0.007	-3.9991	0.9999	-0.0001	-0.0002
			(-4.0)	(1.0)		

Table 6. Identification results for Lorenz equations with 1% noise.

	x_1	x_2	x_3	x_1x_2	x_1x_3	x_2x_3
$\dot{x}_1$	-15.5813	15.9582	0.0757	0.0006	-0.0087	0.0011
	(-16.0)	(16.0)				
$\dot{x}_2$	47.4206	-1.8632	0.0256	0.0034	-1.0252	0.0112
	(45.92)	(-1.0)			(-1.0)	
$\dot{x}_3$	-0.9178	0.5717	-3.9468	0.9902	0.0200	-0.0129
			(-4.0)	(1.0)		

of the least-squares estimation $A\alpha = q$. An obvious source of error is the noise, which is inherent to the system environment, data acquisition and data manipulations. Here, we treat it as external to the system response, and assume it is random and bounded. We have also used the unstable periodic orbits exclusively in our identification scheme for a chaotic system. The unstable periodic orbits are extracted from a chaotic attractor, and used as an approximation of the real periodic orbit of the system. The deviation of the extracted unstable periodic orbits from the real one is another source of error to be discussed in detail.

4.1. Errors Induced by Noise

Suppose noise $n(t)$ is added to a periodic orbit $p(t)$, such that $\tilde{p}(t) = p(t) + n(t)$, where the noise is assumed to be uniformly distributed and uncorrelated to the periodic orbit.

Suppose that the upper bounds of $p(t)$ and $n(t)$ are known, i.e., there exists real positive numbers ψ and ζ such that

$$|p(t)| \leq \psi, \qquad |n(t)| \leq \zeta,$$

for $t \in [0, T]$. We apply this noisy periodic-orbit data to a power-series model (assumed to be valid) nonlinear term x^k. The Fourier coefficients of the noise-contaminated periodic orbit are

$$\tilde{a}_{jk} = \frac{2}{T}\int_0^T \tilde{p}^k(t)\cos(j\omega t)dt$$

$$\tilde{b}_{jk} = \frac{2}{T}\int_0^T \tilde{p}^k(t)\sin(j\omega t)dt,$$

where the subscript k refers to the nonlinear term x^k and j refers to the harmonic of the series.

The differences of the Fourier coefficients between $\tilde{p}(t)$ and $p(t)$ will be bounded by the following relationship:

$$|\Delta a_{jk}| = |a_{jk} - \tilde{a}_{jk}| = \frac{2}{T}|\int_0^T (p^k(t) - \tilde{p}^k(t))\cos(j\omega t)dt|$$

$$= \frac{2}{T}|\int_0^T \sum_{i=1}^{k} \frac{k!}{(k-i)!i!} p^{k-i}(t) n^i(t))\cos(j\omega t)dt|$$

$$\leq 2\sum_{i=1}^{k} \frac{k!}{(k-i)!i!}\psi^k(\frac{\zeta}{\psi})^i$$

for every harmonic j. Normalizing the error bound of a_{jk} by ψ^k yields

$$\frac{|\Delta a_{jk}|}{\psi^k} \leq 2k(\frac{\zeta}{\psi}) + k(k-1)(\frac{\zeta}{\psi})^2 + \cdots. \tag{9}$$

The arguments are the same for $|\Delta b_{jk}| = |b_{jk} - \tilde{b}_{jk}|$.

For $k = 1$, the bound depends on the bound of noise only, which is usually assumed to be as small as a few percent, specified by the noise-to-signal ratio, ζ/ψ. As k increases, the error bound will accumulate, making the perturbations in A larger. If the degree of the nonlinear term in the mathematical model is excessively large, the accuracy of the estimation results will deteriorate rapidly by the noise.

Having these bounds on $|\Delta a_{jk}|$ and $|\Delta b_{jk}|$, we can determine a bound on $\|\Delta A\|$ and $\|\Delta q\|$ due to noise. Then by a method given below, a bound on the norm of the error of the identified parameters, $\|\Delta\alpha\|$, can be estimated.

4.2. Errors Induced by the Periodic-Orbit Extraction

Extracted "periodic orbits" are approximations to the real unstable periodic orbits. Typically, the discrepency is assumed to be small, and the extracted orbits are treated as periodic. In this section, we outline a method of quantifying the error in the extraction.

We had defined recurrent points through a spatial criterion ϵ in the state space, such that $\|x_{i+K} - x_i\| \leq \epsilon$. When this criterion is met, the segment of data from x_i to x_{i+K} is taken as the approximate periodic orbit. This approximation is related to a characteristic quantity of the associated periodic orbit as discussed below.

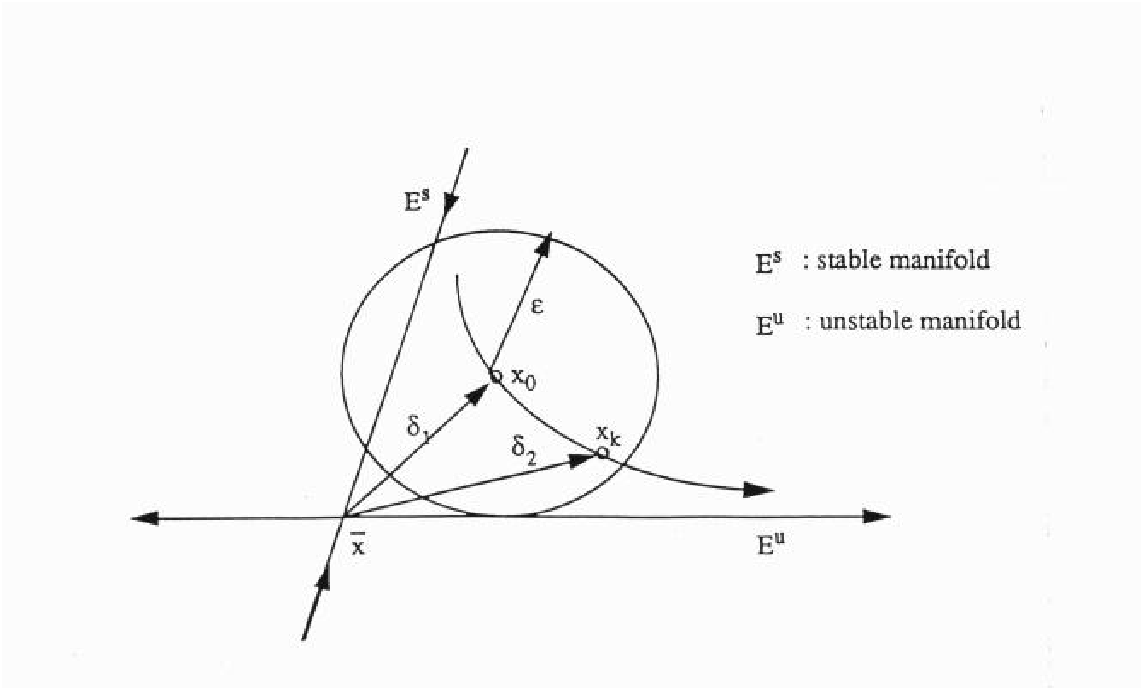

Figure 7. Close look at the periodic orbit extraction on the Poincaré section.

Consider a neighborhood $N(\bar{x})$ of a fixed hyperbolic saddle point $\bar{x}$ of a period-k orbit in the Poincaré section, as sketched in Figure 7.

The dynamics of the system in this neighborhood can be viewed in terms of a linear map T, such that $x_{n+K} - \bar{x} \approx T(x_n - \bar{x})$. Thus, T is the linearized map about the period-k orbit $\bar{x}$ in the period-k Poincaré section, which is invertible, since the periodic point is a saddle for a chaotic system, and the saddle point $\bar{x}$ represents the true periodic orbit of the nonlinear system. If the orbit starting at x_0 is near the saddle to be considered as an approximate periodic orbit of period k, then by the linearized map, the spatial distance between an orbit and its iterate on the period-k Poincaré map can be written as

$$\|x_K - x_0\| = \|(x_K - \bar{x}) - (x_0 - \bar{x})\| = \|T(x_0 - \bar{x}) - (x_0 - \bar{x})\| = \|(T - I)(x_0 - \bar{x})\| \leq \epsilon.$$

Taking the matrix norm, we have

$$\rho_2 \|x_0 - \bar{x}\| \leq \|(T - I)(x_0 - \bar{x})\| \leq \rho_1 \|x_0 - \bar{x}\|,$$

where ρ_1 and ρ_2 are the maximum and minimum singular values of the matrix $T - I$. Since $\|(T - I)(x_0 - \bar{x})\| \leq \epsilon$, the distance between x_0 and $\bar{x}$, by the criterion $\|x_K - x_0\| \leq \epsilon$, is bounded by

$$\delta_1 = \|x_0 - \bar{x}\| \leq \epsilon / \rho_2.$$

The singular value $\rho_2 \neq 0$ since $\bar{x}$ is a hyperbolic saddle point.

Similarly, taking the map backwards, the spatial distance becomes

$$\|x_K - x_0\| = \|(x_K - \bar{x}) - T^{-1}(x_K - \bar{x})\| = \|(T^{-1} - I)(x_K - \bar{x})\|,$$

and the distance between x_K and the saddle is bounded by

$$\delta_2 = \|x_K - \bar{x}\| \leq \epsilon/\mu_2,$$

where μ_2 is the smallest singular value of the matrix $T^{-1} - I$, which is non-zero because $\bar{x}$ is a saddle point. Thus the approximate periodic orbit by the criterion of $\|x_K - x_0\| \leq \epsilon$ is bounded by

$$\delta = \max(\delta_1, \delta_2) \leq \max(\frac{\epsilon}{\rho_2}, \frac{\epsilon}{\mu_2}).$$

The quantity δ bounds the error of the extracted periodic orbit in the period-k Poincaré section. The smaller the pre-specified spatial criterion ϵ is, the smaller the bound δ will be. However, the criterion ϵ should be determined by the data set, since excessively small ϵ may result in no data points fitting in the criterion.

With the error bound δ determined by the characteristic of the periodic orbit as above, we can proceed to bound the errors in A and q, i.e. we can find a bound on $\|\Delta A\|$ and $\|\Delta q\|$ due to the periodic-orbit extraction. The argument is the same as that presented in equation (9), except that the upper bound of noise ζ is replaced by δ, which is an estimate of the error bound under the assumption that the bound on the error in the period-k Poincaré section is representative of (the same order as) that along the entire periodic flow.

In order to apply the above ideas, we have to analyze the data to approximate the map T. The procedure follows ideas of Auerbach *et al.* (1987), Eckmann (1985), Eckmann *et al.* (1986), Kesaraju and Noah (1994) and Sano and Sawada (1985).

Another method of periodic-orbit extraction which is not based on recurrences has been recently proposed (So *et al.*, 1996). This method would not have the same sources of error as the recurrence method used in our study.

4.3. Sensitivity of the Parameter Estimates to Errors

Knowing the error bounds $\|\Delta A\|$ and $\|\Delta q\|$ induced by the noise and by the periodic orbit extraction, we proceed to estimate the sensitivity of the parameter estimation results to these errors.

Let ΔA and Δq be the perturbations of A and q in the linear system, with A being a full-rank matrix. We express the optimal least-squares solution to $A\alpha = q$ as $\alpha = A^{\dagger}q$, and $(A + \Delta A)\tilde{\alpha} = q + \Delta q$ as $\tilde{\alpha} = (A + \Delta A)^{\dagger}(q + \Delta q)$, where the dagger represents the pseudo-inverse. We assume

$$\eta = \max(\frac{\|\Delta A\|}{\|A\|}, \frac{\|\Delta q\|_2}{\|q\|_2}) < \frac{1}{\operatorname{cond} A}, \tag{10}$$

and

$$\sin\theta \equiv \frac{\|r\|_2}{\|q\|_2} < 1, \tag{11}$$

implicitely defining $0 \le \theta < \pi/2$, and where $r = q - A\alpha$ is the residual. Then the error in the estimated parameters α is bounded by (Atkinson, 1989)

$$\frac{\|\tilde{\alpha} - \alpha\|_2}{\|\alpha\|_2} \le \eta(\frac{2\,\mathrm{cond}\,A}{\cos\theta} + \tan\theta(\mathrm{cond}\,A)^2) + 0(\eta^2). \tag{12}$$

For a given model and given periodic orbits, A and q are fixed, and η is determined by $\|\Delta A\|$ and $\|\Delta a\|$ induced by the error sources discussed in the previous sections. If the model is chosen properly, the residual, r, is usually very small, resulting in small θ. Thus the error bound in equation (12) depends linearly on the condition number of A. If the model is not properly chosen, the error bound in equation (12) will depend on the square of $\mathrm{cond}\,A$, significantly increasing the possibility of error. From numerical results presented in Section 3, we found that the choice of a model is an influential factor on estimating the parameters. In our investigations, it seemed that an improperly chosen model would lead to a large condition number for A, resulting in a violation of the assumption in equation (10). In fact, for the example of Section 3.1.3, we directly computed $\|\Delta A\|$ and $\|\Delta q\|$ by comparing the quantities in the noise-free and noisy cases, and found that this assumption was indeed violated. Thus, the estimate (12) may not always be practical, but at least indicates some trends.

4.4. Using Several Periodic Orbits

In this section, we discuss how an application of several periodic orbits together in the identification scheme, as opposed to using a single periodic orbit, can reduce the sensitivity of the parameter estimates to errors.

Suppose a mathematical model has been chosen, and different matrices A_i are formed using p different periodic orbits, such that $i = 1, \ldots, p$. Suppose different sets of parameters $\hat{\alpha}_i$ are then estimated from

$$A_i \hat{\alpha}_i = q_i, \quad i = 1, \ldots, p. \tag{13}$$

Then, combining several A_i into a single matrix A, such that

$$A^T = [A_1^T, A_2^T, \ldots, A_p^T],$$

and combining the corresponding q_i into a single vector q, we obtain a set of parameters $\hat{\alpha}$ from the combined equation, $A\hat{\alpha} = q$, through

$$A^T A \hat{\alpha} = A^T q,$$

which is equivalent to

$$\sum_{j=1}^{p} A_j^T A_j \hat{\alpha} = \sum_{j=1}^{p} A_j^T q_j. \tag{14}$$

Summing equation (13) on i, and subtracting from equation (14), yields

$$\sum_{j=1}^{p} A_j^T A_j (\hat{\alpha} - \hat{\alpha}_j) = \sum_{j=1}^{p} A_j^T A_j ((\hat{\alpha} - \alpha) + (\alpha - \hat{\alpha}_j)) = 0,$$

where α is the true parameter vector. Defining

$$R_i = (A^T A)^{-1}(A_i^T A_i),$$

$$\Delta = \alpha - \hat{\alpha},$$

$$\Delta_i = \alpha_i - \hat{\alpha}_i,$$

we obtain an expression of the error using different sets of periodic orbits, as

$$\Delta = \sum_{j=1}^{p} R_j \Delta_j. \tag{15}$$

Note that $\sum_{j=1}^{p} R_j = I$. Thus, equation (15) implies that the error in parameter estimates from several periodic orbits is a weighted average of parameter errors from individual periodic orbits, through the weights R_j. If a particular periodic orbit were known to yield the best estimation, we would use it for identification. However, we do not know which one will be the best (although it is likely that periodic orbits which tend to "fill" the phase space would sample the vector field better than those that do not). Hence, using a combination of several periodic orbits has advantages. It can give a reasonable estimate, reduce the sensitivity to errors, and improve the statistical properties of the identified results, which has been shown in our previous numerical applications.

5. CONCLUSION

We performed a nonlinear parametric identification on standard chaotic systems. The method involved extracting unstable periodic orbits from the chaotic set, and balancing harmonics on these periodic orbits to identify parameters. We tested the method on forced and autonomous systems, and included a brief noise study in the post-periodic-extraction stage.

The general idea of balancing harmonics to identify parameters had been applied previously by Yasuda and coworkers to systems with periodic responses (Yasuda *et al.*, 1988a; Yasuda *et al.*, 1988b). This work compliments previous work in several ways. It extends the method to chaotic systems. We have also analyzed the effect of noise and outlined ways to put error bounds on the parameter estimates.

We have also outlined a method of estimating the error associated with the extraction of unstable periodic orbits, which is an activity widely seen in the chaos community.

Finally, we saw that chaotic systems are very useful. A single chaotic set provides many unstable periodic solutions. The benefit of having many periodic solutions is that it increases the number of parameters that we can hope to identify. Also, we see that including several period solutions improves the statistics of the parameter estimation, and reduces the possibility of having an estimate based on a poor set of data.

Future studies can involve the application of this method to higher-order chaotic systems. Yasuda *et al.* (1988a, 1988b) and Yasuda and Kamiya (1990) have successfully identified parameters from stable periodic responses of multiple displacements for multi-degree-of-freedom and distributed systems. Our experience suggests that extracted periodic orbits from phase-space reconstructions based on a single observable in higher-order systems may not correspond well to periodic orbits in the real phase space. However, a multiple-observer phase-space reconstruction based on several sensed displacements (Read, 1993) may improve the periodic-orbit extractions.

Studies on experimental systems are in progress. Preliminary results can be found in Yuan (1995). Also, further studies in the error analysis would be fruitful. It would be useful to compute the error quantities outlined in this paper, and gain experience regarding the applicability of the error bounds, both in Fourier coefficients and in extracting periodic orbits. A study on the sensitivity of parameters to sub and superharmonics may indicate which part of the spectrum is most useful.

Acknowledgements. We appreciate the comments of S. L. Chen and H. Khalil. The author CMY was supported by a grant from CSIST of Taiwan.

REFERENCES

Abarbanel, H. D. I., Brown, R., Sidorowich, J., and Tsimring, L., 1993, "The analysis of observed chaotic data in physical systems," *Reviews of Modern Physics* **65**, 1331-1392.

Aguirre, L. A. and Billings, S. A., 1994, "Validating identified nonlinear models with chaotic dynamics," *International Journal of Bifurcation and Chaos* **4**, 109-125.

Atkinson, K. E., 1989, *An Introduction to Numerical Analysis,* second edition, John Wiley and Sons, New York.

Auerbach, D., Cvitanovic, P., Eckmann, J.-P., Gunaratne, G., and Procaccia, I., 1987, "Exploring chaotic motion through periodic orbits," *Physical Review Letters* **58**, 2387-2389.

Cusumano, J. P. and Sharkady, M. T., 1995, "An experimental study of bifurcation, chaos, and dimensionality in a system forced through a bifurcation parameter," *Nonlinear Dynamics* **8**, 467-489.

Eckmann, J.-P., 1985, "Ergodic theroy of chaos and strange attractors," *Reviews of Modern Physics* **57**, 617-656.

Eckmann, J.-P., Kamphorst, S. O., Ruelle, D., and Ciliberto, S., 1986, "Lyapunov exponents from time series," *Physical Review A* **34**, 4971-4979.

Feeny, B. F. and Yuan, C.-M., 1994, "Nonlinear parametric identification by balancing harmonics of extracted periodic orbits," *Fifth Conference on Nonlinear Vibrations, Stability, and Dynamics of Structures,* Blacksburg, VA, June 12-16.

Guckenheimer, J. and Holmes, P. J., 1983, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* Springer-Verlag, New York.

Hammel, S. and Heagy, J., 1992, "Chaotic system identification using linked periodic orbits," *SIAM Conference on Applications of Dynamics Systems,* Snowbird, UT, October 15-19.

Kesaraju, R. V. and Noah, S. T., 1994, "Characterization and detection of parameter variations of nonlinear mechanical systems," *Nonlinear Dynamics* **6**, 433-457.

Lathrop, D. P. and Kostelich, E. J., 1989, "Characterization of an experimental strange attractor by periodic orbits," *Physical Review A* **40**, 4028-4031.

Nayfeh, A. H., 1985, "Parametric identification of nonlinear dynamic systems," *Computers & Structures* **20**, 487-493.

Malasoma, J.-M., Lamarque, C.-H., and Jezequel, L., 1994, "Chaotic behavior of a parametrically excited nonlinear mechanical system," *Nonlinear Dynamics* **5**, 153-160.

Mickens, R. E., 1988, "Bounds on the Fourier coefficients for the periodic solutions of nonlinear oscillator equations," *Journal of Sound and Vibration* **124**, 199-203.

Moon, F. C., 1992, *Chaotic and Fractal Dynamics—An Introduction for Applied Scientists and Engineers,* John Wiley and Sons, New York.

Read, P. L., 1993, "Phase portrait reconstruction using multivariate singular systems analysis," *Physica D* **69**, 353-365.

Sano, M. and Swawda, Y., 1985, "Measurement of the Lyapunov spectrum from a chaotic time series," *Physical Review Letters* **55**, 1082-1085.

So, P., Ott, E., Schiff, S. J., Kaplan, D. T., Sauer, T., and Grebogi, C., 1996, "Detecting unstable periodic orbits in chaotic experimental data," *Physical Review Letters* **76**(25) 4705-4708.

Strang, G., 1976, *Linear Algebra and its Applications,* Academic Press, New York.

Stry, G. and Mook, D. J., 1992, "An experimental study of nonlinear dynamic system identification," *Nonlinear Dynamics* **3**, 1-11.

Tufillaro, N. B., Abbott, T., and Reilly, J., 1992, *An Experimental Approach to Nonlinear Dynamics and Chaos,* Addison-Wesley, Redwood City.

Ueda, Y., 1992, *The Road to Chaos,* Aerial Press, Santa Cruz.

Van de Wouw, N., Verbeek, G., and Van Campen, D. H., 1995, "Nonlinear parametric identification using chaotic data," *Journal of Vibration and Control* **1**, 291-305.

Wolf, A., Swift, J. B., Swinney, H. L., and Vasano, J. A., 1985, "Determining Lyapunov exponents from a time series," *Physica D* **16**, 285-317.

Yasuda, K. and Kamiya, K., 1990, "Identification of a nonlinear beam (proposition of an identification technique)," *JSME International Journal, Series III,*

Yasuda, K. and Kawamura, S., 1989, "A nonparametric identification technique for nonlinear vibratory systems (proposition of a technique)," *JSME International Journal, Series III,* **32**, 365-372.

Yasuda, K., Kawamura, S., and Watanabe, K., 1988a, "Identification of nonlinear multi-degree-of-freedom systems (presentation of an identification technique)," *JSME International Journal, Series III,* **31**, 8-14.

Yasuda, K., Kawamura, S., and Watanabe, K., 1988b, "Identification of nonlinear multi-degree-of-freedom systems (identification under noisy measurements)," *JSME International Journal, Series III,* **31**, 302-309.

Yuan, C.-M., 1995, *A Method of Parametric Identification for Chaotic Systems,* PhD Thesis, Michgan State University, East Lansing.