Discrete Gabor Transform

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Abstract—**The Gabor expansion, which maps the time domain signal into the joint time and frequency domain, has long been recognized as a very useful tool in signal processing. Its applications, however, were limited due to the difficulties associated with selecting the Gabor coefficients. Because time-shifted and frequency-modulated elementary functions in general do not constitute an orthogonal basis, the selections of the Gabor coefficient are not unique. One solution to this problem, developed by Bastiaans, is to introduce an auxiliary biorthogonal function. Then, the Gabor coefficient is computed by the usual inner product rule. Unfortunately, it is not easy to determine the auxiliary biorthogonal function for an arbitrary given synthesis function and sampling pattern. While less success was found in the continuous case, we present a discrete solution in this paper, which is named the discrete Gabor transform (DGT). For a given synthesis window and sampling pattern, computing the auxiliary biorthogonal function of the DGT is nothing more than solving a linear system. The DGT presented applies for both finite as well as infinite sequences. Using the advantages of the nonuniqueness of the auxiliary biorthogonal function at oversampling, we further introduce the so-called orthogonal-like DGT. As the DFT (a discrete realization of the continuous-time Fourier transform), the DGT introduced provides a feasible vehicle to implement the useful Gabor expansion.**

I. Introduction

A HALF century ago, Gabor [7] presented an approach to characterize a time function in time and frequency simultaneously, which later became known as the Gabor expansion. For signal $s(t)$, the Gabor expansion is defined as

$$s(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{m,n} h_{m,n}(t)$$

$$h_{m,n}(t) = h(t - mT) e^{jn\Omega t} \tag{1}$$

where T and Ω represent time and frequency sampling intervals, respectively. The synthesis function $h(t)$ is subject to a unit energy constraint. The existence of (1) has been found to be possible for arbitrary $s(t)$ only for $T\Omega \leq 2\pi$ [2], [9]. $T\Omega = 2\pi$ is called critical sampling and $T\Omega < 2\pi$ is oversampling.

Although the Gabor expansion has been recognized as very useful for signal processing, its applications were limited due to the difficulties associated with computing the Gabor coefficients $C_{m,n}$. According to the Balian–Low theorem, $h_{m,n}(t)$ do not form an orthogonal basis unless the corresponding elementary function $h(t)$ is badly localized in either time or frequency [9]. Therefore, the selection of the Gabor coefficient $C_{m,n}$ in general is not unique. There are two problems that have continued to draw much research—how to define the Gabor coefficients and to what extent the resulting coefficients represent the analyzed signal [1], [2], [7], [8].

One solution to this problem, developed by Bastiaans [2], is to introduce an auxiliary function $\gamma(t)$ and then compute the Gabor coefficient $C_{m,n}$ by the usual inner product rule for projecting $s(t)$ onto $\gamma(t)$, i.e.,

$$C_{m,n} = \int_{-\infty}^{\infty} s(t)\gamma^*_{m,n}(t)\, dt$$

$$\gamma_{m,n}(t) = \gamma(t - mT) e^{jn\Omega t}. \tag{2}$$

Equation (2) is in fact a sampled version of the windowed Fourier transform, which we term the Gabor transform. $\gamma(t)$ in (2) can also be considered as an analysis function.

Substitute (2) into (1). The completeness of (1) leads to the following biorthogonality relationship between $\gamma(t)$ and $h(t)$ [14]:

$$\frac{T_0\Omega_0}{2\pi} \int_{-\infty}^{\infty} h(t)\gamma^*(t - mT_0) e^{-jn\Omega_0 t}\, dt = \delta(m)\delta(n).$$

$$T_0 = \frac{2\pi}{\Omega}, \qquad \Omega_0 = \frac{2\pi}{T}. \tag{3}$$

The major problem of Bastiaan's approach is to compute the analysis function $\gamma(t)$ that satisfies (3) for a given synthesis function $h(t)$ and sampling constants T and Ω. The authors are not aware of the general solution of (3).[1] In fact, even when $\gamma(t)$ can be found in a few special cases [3], [5], [6], [10], $\gamma(t)$ may not be localized. In those cases, the Gabor coefficients $C_{m,n}$ do not reflect signal's local behavior.

While less success was found in continuous cases, the discrete Gabor expansions have been investigated recently [11], [14]. Applying the sampling theory and the discrete Poisson-sum formula to (1)–(3), Wexler and Raz derived the discrete version of the Gabor expansion pair for finite sequence [14]. For an arbitrary given synthesis window and sampling pattern, the implementation of the

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[1]The recently introduced frame approach throws some light on this subject [4]. However, to compute the biorthogonal function, one has to know the frame bounds A and B. Unfortunately, it is not trivial to estimate A and B for an arbitrary given frame.

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finite discrete Gabor expansion, developed by Wexler and Raz, is nothing more than solving a linear system.

In general, the solution of the analysis window function $\tilde{\gamma}(i)$ in the finite discrete Gabor expansion is not unique. In this case, authors solved $\tilde{\gamma}_{\text{opt}}(i)$ which is most similar to the given synthesis window function $\tilde{h}(i)$ [13]. When $\tilde{\gamma}_{\text{opt}}(i)$ is close to $\tilde{h}(i)$, $\tilde{\gamma}_{\text{opt}}(i) \approx \alpha \tilde{h}(i)$ for a constant α, the resulting representation has the same form as the orthogonal representation, despite that $\tilde{h}_{m,n}(i)$ may not be linear independent. The remaining problems in applications are: 1) The analyzed signal is limited to the finite and periodic sequence; 2) The lengths of the analyzed signal and windows have to be equal. Consequently, the memory and computation burden associated with computing the analysis window prohibits processing data of even moderate size.

In this paper, based on the previous work, we develop the discrete Gabor transform (DGT) and Gabor expansion for infinite sequences. In the DGT, the length of the window $h(i)$ and $\gamma(i)$ is independent of the length of the analyzed sequence. Consequently, one can use a finite window to process infinite sequences. The DGT presented applies for both finite as well as infinite sequences.

As in the finite case, the analysis window function of the DGT may also be a solution of an underdetermined linear system. Restricting $\gamma(i)$ to one that is most similar to the given synthesis window $h(i)$, we further obtain the orthogonal-like DGT. In the orthogonal-like DGT, the Gabor coefficient can be thought of as the measure of similarity between the underlying signal $s(i)$ and the individual basis function $h_{m,n}(i)$. Therefore, it will reflect the signal's local behavior as long as the given synthesis window $h(i)$ is indeed localized.

The DGT presented has been extensively tested for various window types and lengths. In all cases, the test results fit perfectly to the theory. As the DFT (discrete Fourier transform), the DGT presented provides a feasible vehicle to implement one of the most important transformations, the Gabor expansion, in signal processing.

The rest of this paper is organized in the following fashion. In Section II, the finite discrete Gabor expansion will be briefly reviewed. In particular, we introduce the orthogonal-like representation. Based on the finite case, the DGT is developed in Section III. Finally, the numerical examples of DGT are provided. Appendix A is devoted to highlight the algorithm of solving the optimal biorthogonal window function $\tilde{\gamma}_{\text{opt}}(i)$. The general biorthogonality relationship will be discussed in Appendix B.

II. Finite Orthogonal-Like Discrete Gabor Expansion

Applying the sampling theory and the discrete Poisson-sum formula, Wexler and Raz obtained a discrete version of the Gabor expansion for the finite and periodic sequence $\tilde{s}(i)$ with a period L [14] as follows:

$$\tilde{s}(i) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} C_{m,n} \tilde{h}_{m,n}(i) \tag{4}$$

$$\begin{aligned} C_{m,n} &= \sum_{i=0}^{L-1} \tilde{s}(i) \tilde{\gamma}^*_{m,n}(i) \\ \tilde{h}_{m,n}(i) &= \tilde{h}(i - m\Delta M) W_L^{n\Delta N i}, \\ \tilde{\gamma}_{m,n}(i) &= \tilde{\gamma}(i - m\Delta M) W_L^{n\Delta N i}, \\ W_L^{n\Delta N i} &= e^{2\pi n \Delta N i/L} \end{aligned} \tag{5}$$

where ΔM and ΔN are time and frequency sampling intervals, respectively. M and N are the numbers of sampling points in time and frequency domains. $\Delta MM = \Delta NN = L$. The condition $\Delta M \Delta N \leq L$ (or $MN \geq L$) must be satisfied for a stable reconstruction. The critical sampling occurs when $\Delta M \Delta N = MN = L$ (The number of coefficients $C_{m,n}$ is equal to the number of time samples $\tilde{s}(i)$). There may be a loss of information in an undersampling condition ($MN < L$). $\tilde{s}(i)$, $\tilde{h}(i)$, and $\tilde{\gamma}(i)$ are periodic in L. In addition, $\tilde{h}(i)$ has unit energy, i.e.,

$$\sum_{i=0}^{L-1} |\tilde{h}(i)|^2 = 1.$$

The Gabor coefficient $C_{m,n}$ in this case is periodic in both m and n, i.e.,

$$C_{m+jM, n+kN} = C_{m,n} \qquad \text{for } j, k = 0, \pm 1, \pm 2, \cdots.$$

Wexler and Raz proved [14] that the biorthogonality between $\tilde{h}(i)$ and $\tilde{\gamma}(i)$ in the finite discrete case is equivalent to

$$\sum_{i=0}^{L-1} \tilde{h}(i + mN) W_L^{-nMi} \tilde{\gamma}^*(i) = \delta(m)\delta(n)$$
$$0 \leq m \leq \Delta N - 1, \qquad \text{and} \quad 0 \leq n \leq \Delta M - 1. \tag{6}$$

Now $\tilde{\gamma}(i)$ becomes a solution of a linear system given by (6). Let $\Delta M \Delta N = p$. Equation (6) can be written in the following matrix form:

$$\begin{aligned} H_{p \times L} \gamma^*_{L \times 1} &= \mu_{p \times 1} \\ \mu_{p \times 1} &= (1, 0, 0, \cdots, 0)^T \end{aligned} \tag{7}$$

where H is a $p \times L$ matrix constructed by

$$H(m\Delta M + n, i) = \tilde{h}(i + mN) W_L^{-nMi}$$
$$0 \leq m < \Delta N, \qquad 0 \leq n \leq \Delta M, \quad \text{and}$$
$$0 \leq i < L.$$

For critical sampling, $\Delta M \Delta N = L$, $\tilde{\gamma}(i)$ is unique if the matrix H is nonsingular. When oversampling, $\Delta M \Delta N = p < L$, the linear system given by (7) is underdetermined and the solution is not unique.

One particularly interesting choice of $\tilde{\gamma}(i)$ is $\tilde{\gamma}_{\text{opt}}$ that is most similar to $\tilde{h}(i)$, e.g.,

$$\Gamma = \min_{\tilde{\gamma}: H\tilde{\gamma}^* = \mu} \left\| \frac{\tilde{\gamma}(i)}{\|\tilde{\gamma}(i)\|} - \tilde{h}(i) \right\|^2 \tag{8}$$

where $\tilde{h}(i)$ has unit energy. If $\tilde{\gamma}(i) \approx \alpha \tilde{h}(i)$, where the constant $\alpha = \|\tilde{\gamma}(i)\|$, then (4) has an orthogonal-like rep-

TABLE I
ERROR Γ

Sampling Pattern	Variance σ^2		
	$0.5\,L/2\pi$	$L/2\pi$	$2\,L/2\pi$
$\Delta M = 16$, $\Delta N = 8$ (critical)	1.2382	0.9494	**0.9002**
$\Delta M = \Delta N = 8$ (double)	0.3035	**0.0865**	0.3035
$\Delta M = 8$, $\Delta N = 4$ (quadruple)	0.3035	0.0612	**0.0037**
$\Delta M = 4$, $\Delta N = 8$ (quadruple)	**0.0037**	0.0612	0.3035

$L = 128$. The boldface numbers correspond to the smallest error for the same sampling pattern but different variances. It is interesting to note that the smallest errors occur when $\sigma^2 = (\Delta M/\Delta N)(L/2\pi)$.

resentation, i.e.,

$$\tilde{s}(i) = \sum_{m,n=0}^{M-1,N-1} \left\{ \sum_{k=0}^{L-1} \tilde{s}(k)\tilde{\gamma}^*_{m,n}(k) \right\} \tilde{h}_{m,n}(i)$$
$$\approx \alpha \sum_{m,n=0}^{M-1,N-1} \left\{ \sum_{k=0}^{L-1} \tilde{s}(k)\tilde{h}^*_{m,n}(k) \right\} \tilde{h}_{m,n}(i). \quad (9)$$

Consequently, the Gabor coefficient $C_{m,n}$ could be considered as the measure of similarity between the underlying signal $\tilde{s}(i)$ and the individual basis function $\tilde{h}_{m,n}(i)$. In this case, Gabor coefficients well reflect signal local behaviors as long as the synthesis window $\tilde{h}(i)$ is localized.

The solution of (8) has been addressed in [13] and will be summarized in Appendix A. If matrix H has a full row rank p, the solution of (8) is the pseudoinverse of H (minimum energy of $\gamma(i)$), i.e.,

$$\gamma^*_{\text{opt}} = H^T(HH^T)^{-1}\mu. \quad (10)$$

The error Γ is a function of the window $\tilde{h}(i)$ as well as the sampling patterns. Table I shows different error values for different sampling patterns and variances, where the window $\tilde{h}(i)$ is a normalized Gaussian function given by

$$\tilde{h}(i) = (\pi\sigma^2)^{-0.25} \exp\left\{ -\frac{[i - 0.5(L-1)]^2}{2\sigma^2} \right\}, \quad \text{for } 0 \le i < L.$$

In general, the error Γ decreases as the oversampling ratio, $L/\Delta M \Delta N$, increases. At the critical sampling, the error Γ is quite large. However, the error Γ could be only 0.0037 at quadruple oversampling. It is important to note that for the same sampling rate, the errors can be quite different depending on the selection of sampling pattern and variance. For instance, at quadruple oversampling (third and fourth rows), using different sampling patterns and variance the corresponding errors range from 0.3035 to 0.0037. At the same oversampling rate, the error Γ usually is smaller when ΔM is proportional to the time duration of the window and ΔN is proportional to the window bandwidth. During our extensive tests, we found that the smallest error occurred when[2]

$$\sigma^2 = \frac{\Delta M}{\Delta N}\frac{L}{2\pi}, \quad \text{for } \sigma \ll L.$$

Once $\gamma(i)$ is determined, it is rather trivial to compute $C_{m,n}$ in (5) by a sampled FFT:

$$\begin{aligned} C_{m,n} &= \sum_{i=0}^{L-1} \tilde{s}(i)\tilde{\gamma}^*(i - m\Delta M)W_N^{-ni} \\ &= \sum_{i=0}^{L-1} R_m(i)W_N^{-ni} \\ &= \sum_{k=0}^{\Delta N-1}\sum_{i=0}^{N-1} R_m(kN+i)W_N^{-n(kN+i)} \\ &= \sum_{k=0}^{\Delta N-1}\sum_{i=0}^{N-1} R_m(kN+i)W_N^{-ni}, \end{aligned}$$

where the second summation is an N-point FFT.

III. Discrete Gabor Expansion for Infinite Sequences

The finite discrete Gabor expansion presented in Section II works very well when the length of analyzed sequences is small. For an arbitrary given synthesis window $\tilde{h}(i)$ and sampling pattern one can readily obtain $\tilde{\gamma}(i)_{\text{opt}}$ if matrix H is of a full row rank. The authors are not aware of other existing techniques that achieve the same result.

In many real applications, however, the number of samples could be very large. In those cases, the algorithm introduced earlier is no longer adequate. It is desirable to investigate the discrete version of the Gabor expansion for infinite sequences.

Given finite signal $s(i)$ with length L_s and synthesis window $h(i)$ with length L, let us construct two periodic sequences $\hat{s}(i)$ and $\hat{h}(i)$ with period $L0 = L + L_s$:

$$\hat{s}(i) = \hat{s}(i + kL0) = \begin{cases} 0 & -L \le i < 0 \\ s(i) & 0 \le i < L_s \end{cases}$$
$$\hat{h}(i) = \hat{h}(i + kL0) = \begin{cases} h(i) & 0 \le i < L \\ 0 & L \le i < L0 \end{cases}$$
$$k = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

$\hat{s}(i)$ is the original sequence $s(i)$ padded with L points of zero in the beginning and $\hat{h}(i)$ is an L-point window function $h(i)$ appended with L_s points of zero in the end.

Now, let us compute the finite discrete Gabor expansion of $\hat{s}(i)$ by the window $\hat{h}(i)$. In this case, the number of frequency sampling points $N = L0/\Delta N$, where ΔN denotes the frequency sampling constant. Shifting the range of indices m and i in (4), (5) yields the following

[2]A theoretical proof has not yet been found. However, we feel that this interesting observation is worth presenting as a guideline in selecting the variance and sampling constants.

finite discrete Gabor expansion pair:

$$\hat{s}(i) = \sum_{m=-L/\Delta M}^{Ls/\Delta M - 1} \sum_{n=0}^{N-1} C_{m,n} \hat{h}_{m,n}(i),$$

$$C_{m,n} = \sum_{i=-L}^{Ls-1} \hat{s}(i)\hat{\gamma}^*_{m,n}(i) = \sum_{i=0}^{Ls-1} \hat{s}(i)\hat{\gamma}^*_{m,n}(i). \quad (11)$$

$$\hat{h}_{m,n}(i) = \hat{h}(i - m\Delta M) W_N^{ni}$$

$$\hat{\gamma}_{m,n}(i) = \hat{\gamma}(i - m\Delta M) W_N^{ni} \quad (12)$$

where we use the fact that $\Delta N N = L0$ and $\hat{s}(i) = 0$ for $i < 0$. The necessary condition for a stable reconstruction is

$$\Delta M \Delta N = \Delta M \frac{L0}{N} \le L0$$

that is, $N/\Delta M \ge 1$.

From (6), the biorthogonality relationship between $\hat{h}(i)$ and $\hat{\gamma}(i)$ is

$$\sum_{i=0}^{L0-1} \hat{h}(i + mN) W_{\Delta M}^{-ni} \hat{\gamma}^*(i) = \delta(m)\delta(n)$$

$$0 \le m < \frac{L0}{N} \quad \text{and} \quad 0 \le n < \Delta M \quad (13)$$

where $\hat{\gamma}(i)$ is also periodic with period $L0$. It is natural to define the length of $\gamma(i)$ to be equal to that of the given synthesis window $h(i)$, i.e.,

$$\hat{\gamma}(i) = \hat{\gamma}(i + kL0) = \begin{cases} \gamma(i) & 0 \le i < L \\ 0 & L \le i < L0 \end{cases}$$

$$k = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

As shown in Appendix B, under this assumption we can rewrite (13) as

$$\sum_{i=0}^{L-1} \bar{h}(i + mN) W_{\Delta M}^{-ni} \gamma^*(i) = \delta(m)\delta(n)$$

$$0 \le m < \frac{2L}{N} - 1, \quad \text{and} \quad 0 \le n < \Delta M \quad (14)$$

where the periodic sequence $\bar{h}(i)$ with period $2L - N$ is given by

$$\bar{h}(i) = \bar{h}(i + k[2L - N]) = \begin{cases} h(i) & 0 \le i < L \\ 0 & L \le i < 2L - N \end{cases}$$

$$k = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

It is worthwhile to note that $\gamma(i)$ obtained from (14) is a special solution of (13), whose length is equal to the length of the given window function $h(i)$. The significance of (14) is its independence of the signal length L_s, which suggests that $\gamma(i)$ can be completely determined regardless of the analyzed data size. Equation (14) can also be written in the matrix form, i.e.,

$$\bar{H}\gamma^* = \bar{\mu}.$$

$$\bar{\mu} = (1, 0, 0, \cdots, 0)^T \quad (15)$$

where $\bar{H}$ is $((2\Delta M/N)L - \Delta M) \times L$ matrix constructed by

$$\bar{H}(m\Delta M + n, i) = \bar{h}(i + mN) W_{\Delta M}^{-ni}$$

$$0 \le m < \frac{2L}{N} - 1, \quad 0 \le n < \Delta M, \quad \text{and}$$

$$0 \le I < L.$$

Fig. 1 depicts $\hat{s}(i)$ and $\hat{\gamma}(i)$. Due to sufficient zero padding in both $\hat{s}(i)$ and $\hat{\gamma}(i)$, the Gabor coefficients $C_{m,n}$ can be completely determined without rolling over, which actually releases the periodic constraint.

With L remaining finite, letting $L_s \to \infty$ and thereby $L0 \to \infty$, (11), (12) directly lead to the discrete version of the Gabor expansion pair for infinite sequence $s(i)$, i.e.,

$$s(i) = \sum_{m=-L/\Delta M}^{\infty} \sum_{n=0}^{N-1} C_{m,n} h(i - m\Delta M) W_N^{ni} \quad (16)$$

$$C_{m,n} = \sum_{i=0}^{\infty} s(i)\gamma^*(i - m\Delta M) W_N^{-ni}. \quad (17)$$

To distinguish it from other related approaches [1], [11], [14], we name (17) the discrete Gabor transform (DGT) and (16) the discrete Gabor expansion (or inverse DGT).

If we restrict the length of $\gamma(i)$ and $h(i)$ to be the same, then $\gamma(i)$ is determined by (15). At critical sampling, $N/\Delta M = 1$, (15) is an overdetermined linear system. The solution usually does not exist since $\bar{\mu}$ must be an element of range of $\bar{H}$. At oversampling, $N/\Delta M \ge 2$, (15) is an underdetermined linear system. In this case, the system either has no solution or has an infinite number of solutions. The particularly interesting choice of $\gamma(i)$ is one that is most similar to the given $h(i)$ since such constraint will lead to a useful orthogonal-like representation.

In fact, (14) and (6) have a similar form. Analog to the finite case [13], the $\gamma(i)$ that is most similar to $h(i)$, in the sense of least square error, can be easily obtained as[3]

$$\gamma^*_{\text{opt}} = \bar{H}^T(\bar{H}\bar{H}^T)^{-1}\bar{\mu}. \quad (18)$$

When $\gamma_{\text{opt}}(i)$ is close to $h(i)$, (16) can be thought of as orthogonal-like expansion.

One of the main purposes of applying the Gabor expansion is to have a localized synthesis function as well as the localized Gabor coefficients. The Zak transform approach [1] presents many interesting features, but generally it is not known to what extent the resulting coefficients represent the analyzed signal. On the other hand, the orthogonal-like DGT presented has meaningful physical interpretations. As long as the synthesis window function $h(i)$ is localized, the orthogonal-like DGT will

[3]Equation (18) only holds for $\bar{H}$ with full row rank. During our continuous research, we have solved $\gamma^*_{\text{opt}}(i)$ for more general cases [12].

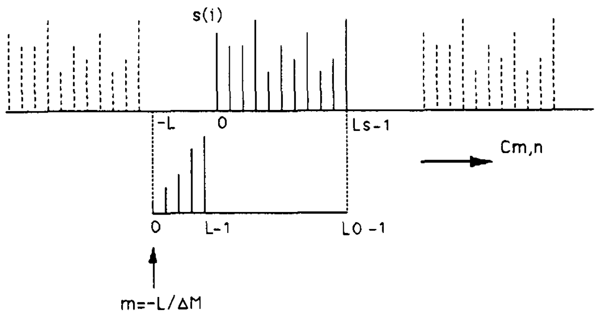

Fig. 1. $\hat{s}(i)$ and $\hat{\gamma}(i)$.

well reflect signal local behaviors because $C_{m,n}$ is very close to the inner product of $s(i)$ and $h_{m,n}(i)$.

Analog to the finite case, once $\gamma_{\text{opt}}(i)$ is found $C_{m,n}$ can be efficiently evaluated by a sampled FFT. Since $\gamma(k) = 0$ for $k < 0$ and $k \geq L$. Substituting $k = i - m\Delta M$ in (17) yields

$$
\begin{aligned}
C_{m,n} &= \sum_{k=0}^{L-1} s(k + m\Delta M)\gamma^*(k)W_N^{-n(k+m\Delta M)} \\
&= W_N^{-nm\Delta M} \sum_{k=0}^{L-1} R_m(k)W_N^{-nk} \\
&= W_N^{-nm\Delta M} \sum_{k=0}^{\Delta N-1} \sum_{i=0}^{N-1} R_m(kN + i)W_N^{-ni}.
\end{aligned}
$$

The right-side summation is a standard N-point FFT.

Figs. 2 to 5 illustrate biorthogonal analysis windows corresponding to the Gaussian function, the chirp function, the one-side exponential function, and a smoothed exponential function. The similarity between $\gamma_{\text{opt}}(i)$ and $h(i)$ usually is proportional to the oversampling rate. At the same oversampling rate, in general, the error Γ is smaller for the smoothed synthesis window, as indicated in Figs. 4 and 5. In all our simulations, reconstruction errors (MSE) were around 10^{-15}, which were virtually error-free reconstruction.

IV. Conclusions

In this paper, we present a feasible algorithm to implement the Gabor expansion, whose coefficients are computed by the discrete Gabor transform (DGT). We begin with the finite discrete case, in which the analysis function $\tilde{\gamma}(i)$ is simply a solution of a linear system determined by a given synthesis function $\tilde{h}(i)$ as well as the sampling pattern. In order to have localized Gabor coefficients $C_{m,n}$, we restrict ourselves to $\tilde{\gamma}_{\text{opt}}(i)$ which is most similar to the given synthesis window function $\tilde{h}(i)$, and thereby obtain the so-called orthogonal-like expansion. Based on the finite discrete Gabor expansion, we developed DGT which applies for both finite as well as infinite sequences. In the DGT, the selection of the window length is independent of the size of the analyzed data. For an arbitrary given synthesis window and sampling pattern, we are always able to find the corresponding biorthogonal

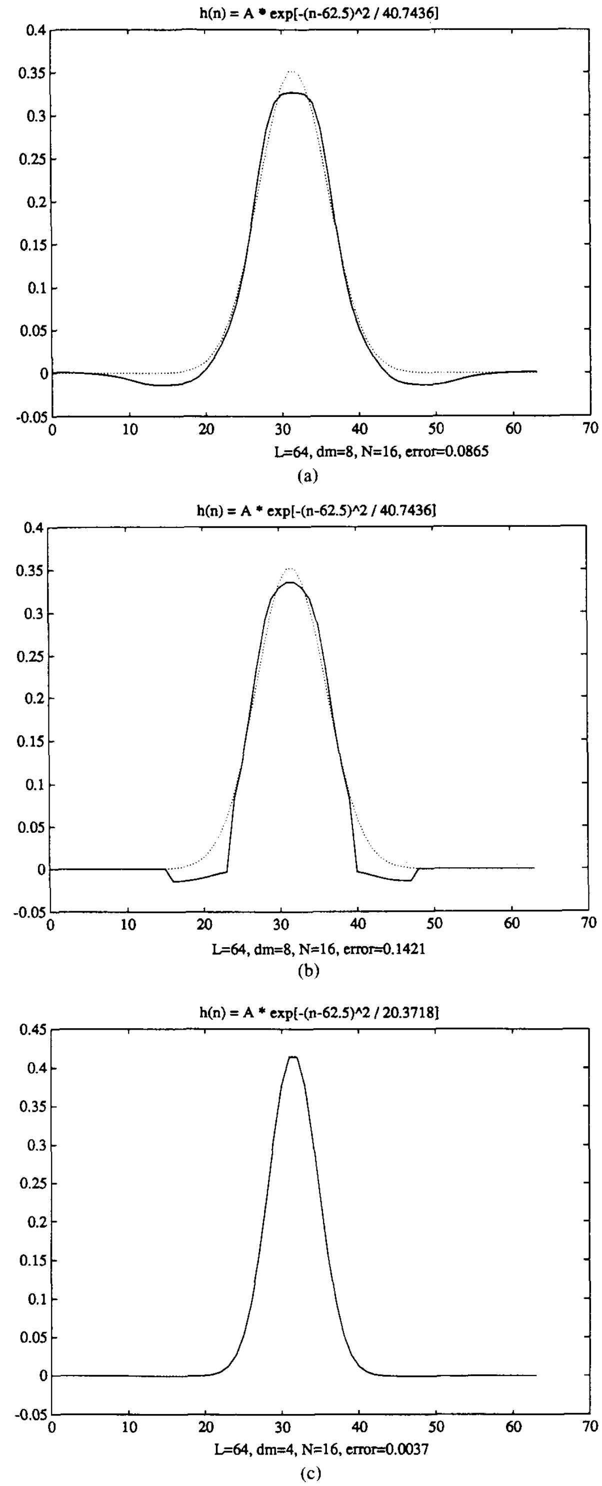

Fig. 2. Normalized Gaussian basis and corresponding biorthogonal window. (Solid lines represent the given synthesis window $h(i)$. Broken line curves represent $\gamma_{\text{opt}}(i)$. Oversampling rate $= N/dm$.) (a) Finite discrete Gabor expansion at double oversampling. (b) Infinite discrete Gabor expansion (DGT) at double oversampling. (c) DGT at quadruple oversampling. (The given synthesis window and the optimal analysis window virtually are identical.)

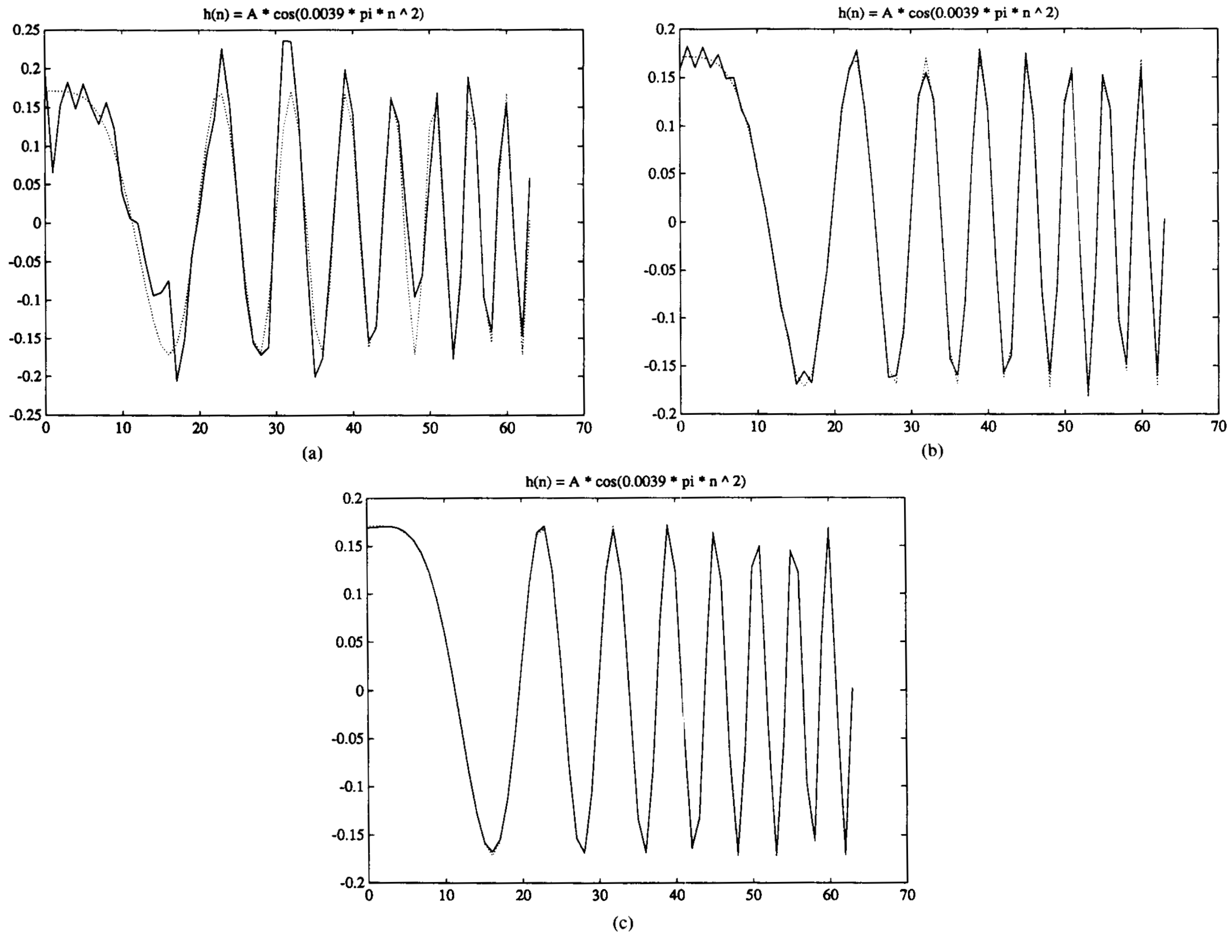

Fig. 3. Chirp basis and biorthogonal window DGT. As the oversampling rate increases, the optimal analysis window gets closer to the given synthesis window. (Solid lines represent the given synthesis window $h(i)$. Broken line curves represent $\gamma_{opt}(i)$. Oversampling rate $= N/dm$.) (a) $L = 64$, $dm = 2$, $N = 8$, error $= 0.3031$. (b) $L = 64$, $dm = 2$, $N = 16$, error $= 0.061$. (c) $L = 64$, $dm = 1$, $N = 16$, error $= 0.0123$.

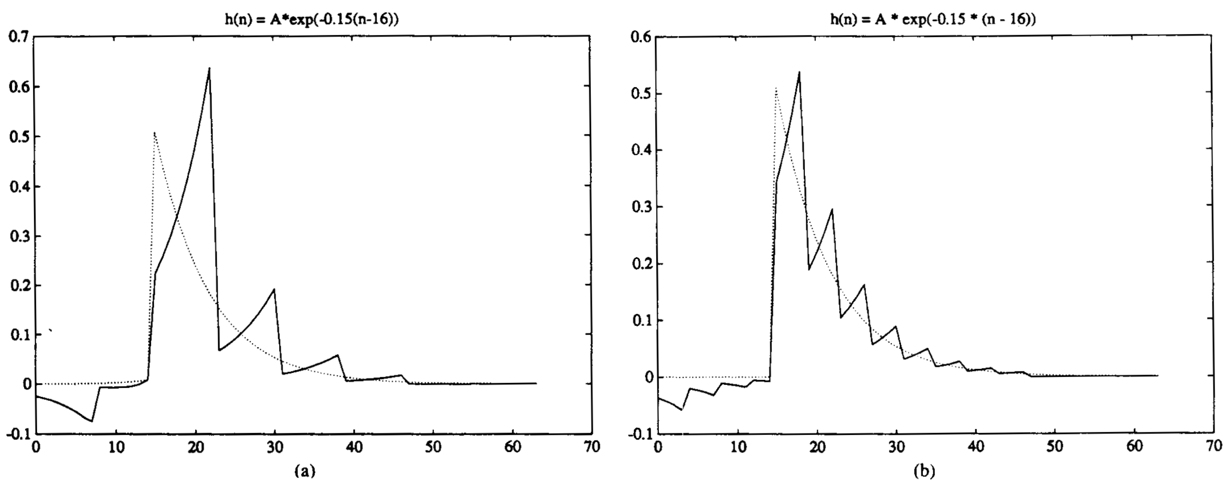

Fig. 4. One-side exponential basis and biorthogonal window of DGT. (Solid lines represent the given synthesis window $h(i)$. Broken line curves represent $\gamma_{opt}(i)$. Oversampling rate $= N/dm$.) (a) $L = 64$, $dm = 8$, $N = 32$, error $= 0.6436$. (b) $L = 64$, $dm = 4$, $N = 32$, error $= 0.3444$. (c) $L = 64$, $dm = 2$, $N = 32$, error $= 0.1824$. (d) $L = 64$, $dm = 1$, $N = 32$, error $= 0.1052$.

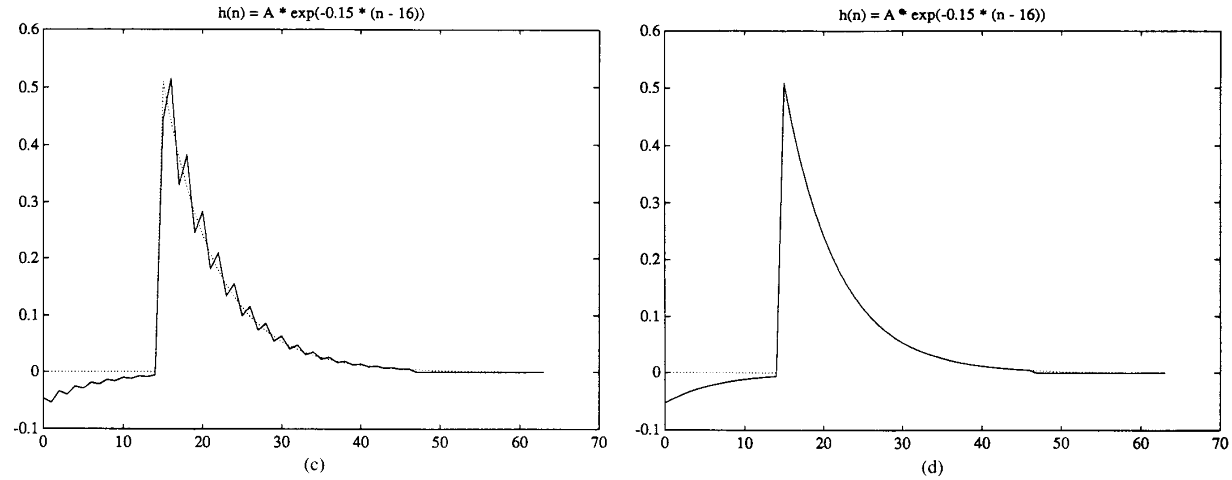

Fig. 4. (*Continued*)

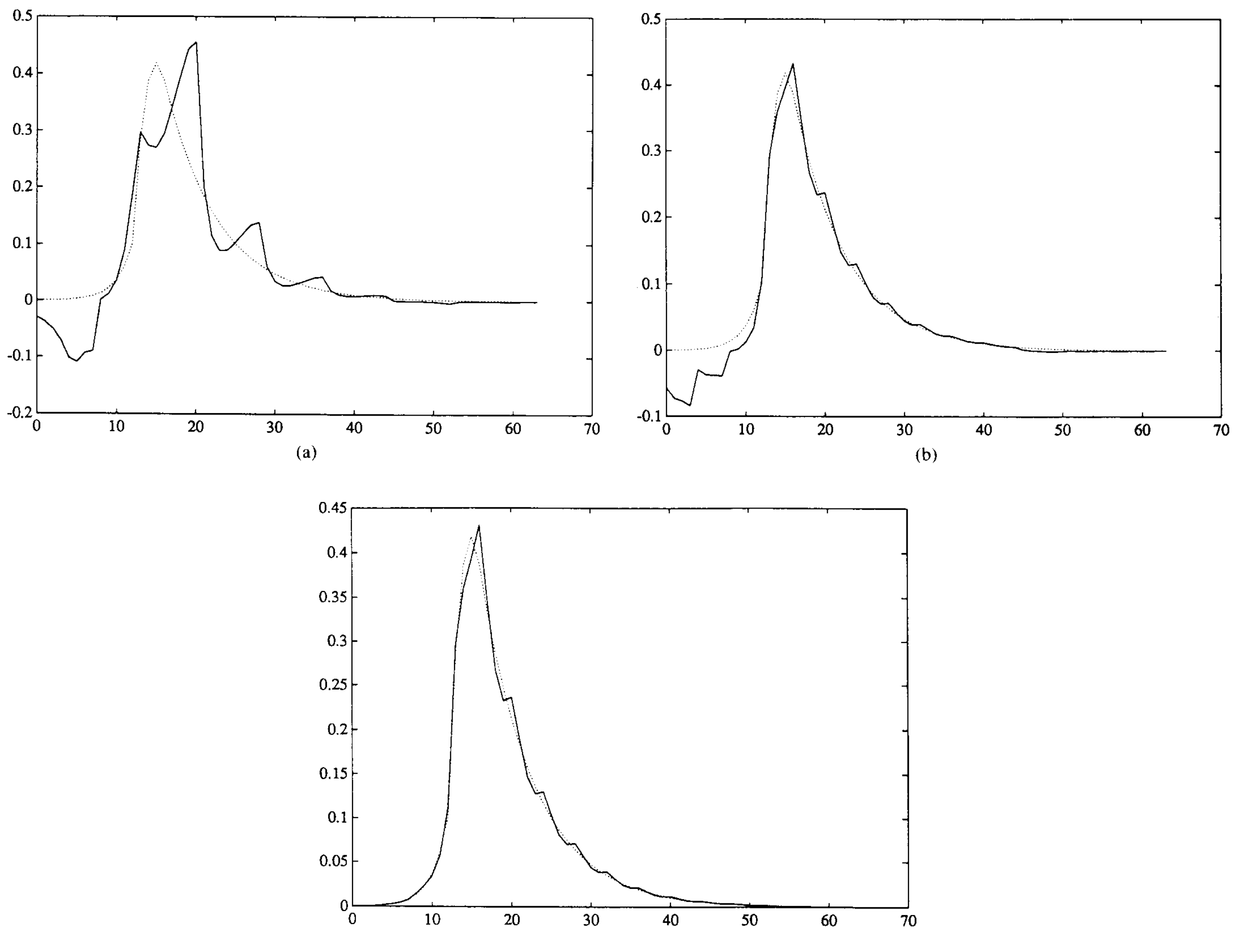

Fig. 5. Smoothed one-side exponential basis and biorthogonal window of DGT. As shown in Figs. 4 and 5, at the same oversampling rate the optimal analysis window in general is closer to the synthesis window if it is smoother. (Solid lines represent the given synthesis window $h(i)$. Broken line curves represent $\gamma_{opt}(i)$. Oversampling rate $= N/dm$.) (a) $L = 64$, $dm = 8$, $N = 32$, error $= 0.4435$. (b) $L = 64$, $dm = 4$, $N = 32$, error $= 0.1875$. (c) $L = 64$, $dm = 4$, $N = 64$, error $= 0.0736$.

analysis window (if it exists). The implementation of the DGT is nothing more than solving a linear system. Analog to the finite case, we further discussed the orthogonal-like DGT. As the DFT (a discrete realization of the continuous-time Fourier transform), similarly, the DGT presented in this paper provides a feasible vehicle to realize the useful Gabor expansion.

Appendix A

If H has a full row rank p, then we can write H in terms of the QR decomposition, i.e.,

$$H^T_{L\times p} = Q_{L\times L}\left|\frac{R_{p\times p}}{O_{(L-p)\times p}}\right| \tag{A.1}$$

where Q is orthonormal and R is upper triangular. Because the first row of H is h^T where $h^T = [h(0), h(1), \cdots, h(L-1)]$, (A.1) leads to

$$|h, \cdots| = |r_{1,1}q_1, r_{1,2}q_1 + r_{2,2}q_2, \cdots|.$$

Hence

$$h = r_{1,1}q_1 \tag{A.2}$$

where q_1 is the first column of Q. Substituting (A.1) into

$$H\gamma = \mu \tag{A.3}$$

obtain

$$|R^T|O|\,Q^T\gamma = |R^T|O|\left|\frac{x_{p\times 1}}{y_{(L-p)\times 1}}\right| = \mu_{p\times 1} \tag{A.4}$$

where

$$Q^T\gamma = \left|\frac{x_{p\times 1}}{y_{(L-p)\times 1}}\right|.$$

Since $Q^TQ = I$, we could further have

$$\gamma = Q\left|\frac{x_{p\times 1}}{y_{(L-p)\times 1}}\right| = |Q_x\ Q_y|\left|\frac{x_{p\times 1}}{y_{(L-p)\times 1}}\right|$$

$$= Q_x x + Q_y y. \tag{A.5}$$

From (A.4),

$$x = (R^T)^{-1}\mu. \tag{A.6}$$

Several remarks are now in order.

a) Because $h = r_{1,1}q_1$, the window function $h(i)$ is in the range of the matrix Q_x. Therefore,

$$Q_y^T h = 0. \tag{A.7}$$

b) The biorthogonal analysis window function γ is the sum of two orthogonal vectors, $Q_x x + Q_y y$. Thus

$$\|\gamma\|^2 = \|x\|^2 + \|y\|^2. \tag{A.8}$$

c) While $Q_x x$ is determined by the given matrix H as well as the biorthogonality relationship, $Q_y y$ depends on the particular constraints. For $Q_y y = 0$, $\gamma = Q_x x = \gamma_{\min}$, which has a minimum energy.

Rewriting the error equation obtain

$$\Gamma = \min_{\gamma: H\gamma=\mu}\left\|\frac{\gamma(i)}{\|\gamma(i)\|} - h(i)\right\|^2 = 2\left(1 - \frac{\mathrm{Re}\,(\gamma^T h)}{\|\gamma\|}\right) \tag{A.9}$$

then minimizing Γ w.r.t. γ is equivalent to

$$\max_{\gamma: H\gamma=\mu}\ \xi = \max_{\gamma: H\gamma=\mu}\frac{\mathrm{Re}\,(\gamma^T h)}{\|\gamma\|}. \tag{A.10}$$

Replacing γ and $\|\gamma\|$ via (A.5) and (A.8), (A.10) can be written as

$$\max_{\gamma: H\gamma=\mu}\ \xi = \max_{y\in R^{L-p}}\frac{\mathrm{Re}\,(x^TQ_x^Th)}{\sqrt{\|x\|^2 + \|y\|^2}} \tag{A.11}$$

where we use the fact that $Q_y h = 0$. Obviously, the maximum of ξ occurs when $\|y\|$ is a zero vector. Substituting $y = 0$ in (A.5) yields

$$\gamma_{\mathrm{opt}} = Q_x x = Q_x(R^T)^{-1}\mu. \tag{A.12}$$

The corresponding least square error is

$$\Gamma = 2\left(1 - \frac{\mathrm{Re}\,(\mu^TR^{-1}Q_x^Th)}{\|(R^T)^{-1}\mu\|}\right). \tag{A.13}$$

γ_{opt} obtained via (A.12) implies $\|y\| = 0$, which says that γ_{opt} contains the minimum energy. Hence, γ_{opt} is also a pseudoinverse of matrix H, i.e.,

$$\gamma_{\mathrm{opt}} = H^T(HH^T)^{-1}\mu. \tag{A.14}$$

By $H^T = QR$, one can easily verify that (A.14) and (A.12) are identical.

Appendix B

First let us assume that

1) $L = N\Delta N_L$, where ΔN_L is an integer;
2) $L0 = N\Delta N_{L0}$.

Then, (13) can be rewritten as

$$\sum_{k=0}^{\Delta N_L - 1}\sum_{i=0}^{N-1}\hat{h}((m+k)N + i)W_{\Delta M}^{-ni}\hat{\gamma}^*(kN+i)$$
$$= \delta(m)\delta(n) \qquad 0 \le m \le \Delta N_{L0},$$
$$0 \le n < \Delta M, \quad \text{and} \quad 0 \le i < N. \tag{B.1}$$

Equation (B.1) can also be written in the matrix form

$$\hat{H}\gamma^* = \hat{\mu}$$
$$\hat{\mu} = (1, 0, 0, \cdots, 0)^T \tag{B.2}$$

where

$$\hat{H} = \left| \begin{array}{cccccccc} B_0 & B_1 & \cdots & B_{\Delta N_L - 1} & B_{\Delta N_L} & \cdots & B_{\Delta N_{L0} - 2} & B_{\Delta N_{L0} - 1} \\ B_1 & B_2 & \cdots & B_{\Delta N_L} & B_{\Delta N_L + 1} & \cdots & B_{\Delta N_{L0} - 1} & B_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{\Delta N_{L0} - 1} & B_0 & \cdots & \cdots & \cdots & \cdots & \cdots & B_{\Delta N_{L0} - 2} \end{array} \right| \tag{B.3}$$

where B_{m+k} is $\Delta M \times N$ matrix with element

$$B_{m+k}[i, n] = \hat{h}((m + k)N + i) W_{\Delta M}^{-ni} \qquad 0 \leq n \leq \Delta M \quad \text{and} \quad 0 \leq i < N. \tag{B.4}$$

Because $\hat{h}(i)$ is periodic with period $L0$, $B_{m+k} = B_{m+k+L0}$. Consequently, each row of $\hat{H}$ is a circular shifted version of the previous row. Moreover, $B_{m+k} = 0$ for $m + k \geq \Delta N_L$ since $\hat{h}(i) = 0$ for $L \leq i < L0$. Replacing $B_{m+k} = 0$ for $m + k \geq \Delta N_L$, $\hat{H}$ can be rewritten as

$$\hat{H} = \left| \begin{array}{ccccccc} B_0 & B_1 & \cdots & B_{\Delta N_L - 1} & \cdots & 0 & 0 \\ B_1 & B_2 & \cdots & 0 & \cdots & 0 & B_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{\Delta N_L - 1} & 0 & \cdots & 0 & \cdots & \cdots & B_{\Delta N_L - 2} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & B_{\Delta N_L - 1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & B_0 & B_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & B_0 & \cdots & B_{\Delta N_L - 2} & 0 & \cdots & 0 \end{array} \right|. \tag{B.5}$$

Rearranging the row of $\hat{H}$ in (B.5) such that the last $\Delta N_L - 1$ rows are immediately after the ΔN_Lth row, we obtain

$$\hat{H} = \left| \begin{array}{cccccccc} B_0 & B_1 & \cdots & B_{\Delta N_L - 1} & 0 & \cdots & 0 & 0 \\ B_1 & B_2 & \cdots & 0 & 0 & \cdots & 0 & B_0 \\ B_{\Delta N_L - 1} & 0 & \cdots & 0 & 0 & \cdots & \cdots & B_{\Delta N_L - 2} \\ 0 & 0 & \cdots & B_0 & B_1 & B_2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & B_0 & \cdots & B_{\Delta N_L - 2} & B_{\Delta N_L - 1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & & & & \end{array} \right|. \tag{B.6}$$

For the first ΔN_L columns of $\hat{H}$, the last $\Delta N_{L0} - (2\Delta N_L - 1)$ rows are zero. Group $\hat{H}$ in (B.6) into submatrices such that

$$\hat{H} = \left| \begin{array}{cc} A_0 & A_1 \\ 0 & A_3 \end{array} \right| \tag{B.7}$$

where

$$A_0 = \left| \begin{array}{cccc} B_0 & B_1 & \cdots & B_{\Delta N_L - 1} \\ B_1 & B_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ B_{\Delta N_L - 1} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & B_1 & \cdots & B_{\Delta N_L - 2} \end{array} \right|.$$

Since the dimension of B_{m+k} is $\Delta M \times N$, A_0 is $\Delta M(2\Delta N_L - 1) \times L$, that is, $((2\Delta M/N)L - \Delta M) \times L$, matrix. For a stable reconstruction, $\Delta M/N \leq 1$. Replacing $\hat{H}$ in (B.2) obtains

$$\begin{vmatrix} A_0 & A_1 \\ 0 & A_3 \end{vmatrix} \begin{vmatrix} \gamma_{L \times 1} \\ z_{(L0-1) \times 1} \end{vmatrix} = \hat{\mu}. \tag{B.8}$$

It is natural to have the lengths of the given synthesis window and the analysis window equal. In this case, the solution of (B.8) is

$$\begin{cases} A_0 \gamma^* = \mu \\ z = 0 \end{cases}$$
$$\mu = (1, 0, 0, \cdots, 0)^T \tag{B.9}$$

Equation (B.9) can be explicitly written as

$$\sum_{i=0}^{L-1} \bar{h}(i + mN) W_{\Delta M}^{-ni} \gamma^*(i) = \delta(m)\delta(n).$$
$$0 \leq m < \frac{2L}{N} - 1, \quad \text{and} \quad 0 \leq n < \Delta M \tag{B.10}$$

where the periodic sequence $\bar{h}(i)$ with period $2L - N$ is given by

$$\bar{h}(i) = \bar{h}(i + k[2L - N])$$
$$= \begin{cases} h(i) & 0 \leq i < L \\ 0 & L \leq i < 2L - N \\ & k = 0, \pm 1, \pm 2 \cdots. \end{cases}$$

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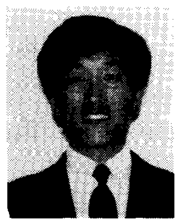

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