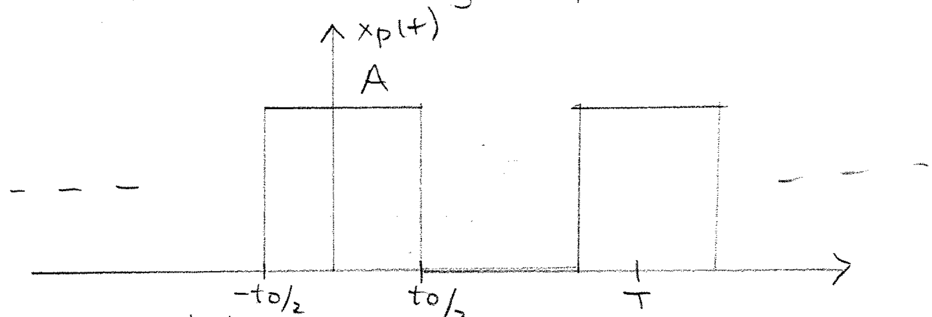
Example: Connections Between F.S. and F.T

Consider the periodic rectangular pulse train



$$X[k] = \frac{1}{T} \int_{-t_0/2}^{t_0/2} A\, e^{-j2\pi k f_0 t}\, dt \qquad f_0 = \frac{1}{T}$$

$$= \frac{1}{T} \int_{-t_0/2}^{t_0/2} A\, e^{-j2\pi \frac{kt}{T}}\, dt$$

$$= \frac{A}{T} \left. \frac{e^{-j2\pi \frac{kt}{T}}}{\left(-j\frac{2\pi k}{T}\right)} \right|_{-t_0/2}^{t_0/2}$$

$$= \frac{A}{-j2\pi k} \left(e^{-j\frac{\cancel{2}\pi k t_0}{\cancel{2}T}} - e^{\frac{j\cancel{2}\pi k t_0}{\cancel{2}T}} \right)$$

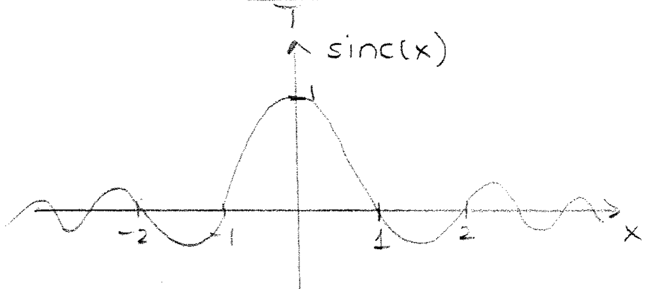
$$= \frac{A}{\pi k} \left(\frac{e^{\frac{j\pi k t_0}{T}} - e^{-\frac{j\pi k t_0}{T}}}{2j} \right)$$

$$= \frac{A}{\pi k} \sin\left(\frac{\pi k t_0}{T}\right)$$

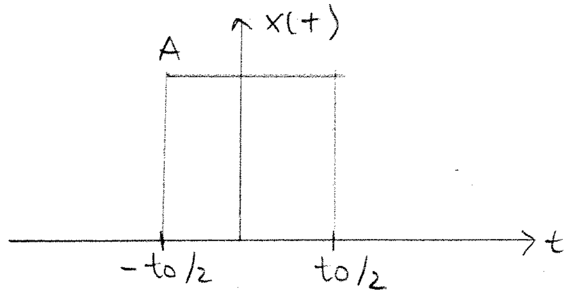
$$A t_0 f_0 \frac{\sin(\pi k t_0 f_0)}{\pi k t_0 f_0}$$

Note: $\frac{\sin(\pi x)}{\pi x} = \mathrm{sinc}(x)$

$$X[k] = \frac{A t_0}{T} \,\mathrm{sinc}(k f_0 t_0)$$



Now if we have the aperiodic signal $x(t)$ $(T \to \infty)$

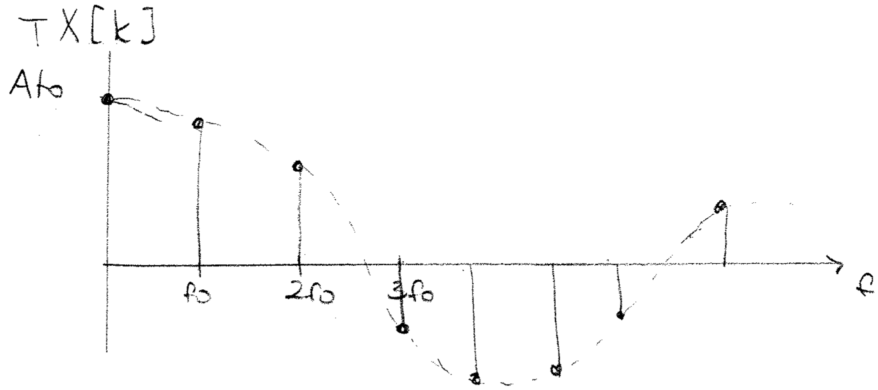


$$X(f) = T X[k] = \lim_{T \to \infty} A t_0 \operatorname{sinc}(k f_0 t_0)$$

$$k f_0 \to f$$

$$X(f) = A t_0 \operatorname{sinc}(f t_0)$$

What's the difference?



As you increase T you sample the sinc envelope more often
as $T \to \infty$ you get the continuous spectrum $(f_0 \to df \to 0)$