ECE 360 EXAM 2
November 20, 2002

- No textbooks, notes or HW solutions.
- One page of hand-written notes.
- Calculators are allowed.
- Exam is 50 minutes.
- To maximize your score on this exam, read the questions carefully and write legibly. For those problems that allow partial credit, show your work clearly.
- Good luck.

Part A- Multiple choice and short answer questions. It is not necessary to show work, no partial credit will be given.

1. [5] A continuous-time LTI system is described by the corresponding input/output differential equation

\[ \frac{d^2 y(t)}{dt^2} - 4 \frac{dy(t)}{dt} + 3y(t) = 4 \frac{dx(t)}{dt} - x(t) \]

Which one of the following statements is true for this system?

a) \( H(s) = \frac{s^2 - 4s + 3}{4s - 1} \), and the system is stable.

b) \( H(s) = \frac{s^2 - 4s + 3}{4s - 1} \), and the system is unstable.

c) \( H(s) = \frac{4s - 1}{s^2 - 4s + 3} \), and the system is stable.

d) \( H(s) = \frac{4s - 1}{s^2 - 4s + 3} \), and the system is unstable

2. [5] What is the Fourier transform of the periodic signal \( x(t) = \sum_{k=-\infty}^{\infty} rect(t - 2k) \)?

a) \( X(f) = sinc(f) \)

b) \( X(f) = sinc(f)e^{-j4\pi f} \)

c) \( X(f) = \sum_{k=-\infty}^{\infty} 0.5sinc(0.5k)\delta(f - 0.5k) \)

d) \( X(f) = \sum_{k=-\infty}^{\infty} 2sinc(2k)\delta(f - 2k) \)
3. [5] The Fourier transform of \( x(t) = |t|e^{-|t|} \) is

\[
X(f) = \frac{2 + 8\pi^2 f^2}{1 - 8\pi^2 f^2 + 16\pi^4 f^4}
\]

Based on this, what is the inverse Fourier transform of \( X(f) = |f|e^{-|f|} \)?

Hint: If you use one of the properties discussed in class the answer is immediate and requires no computation.

\[ \text{a)} \ x(t) = \frac{2 + 8\pi^2 t^2}{1 - 8\pi^2 t^2 + 16\pi^4 t^4} \]
\[ \text{b)} \ x(t) = |t|e^{-|t|} \]
\[ \text{c)} \ x(t) = e^{-|t|}u(t) \]
\[ \text{d)} \ x(t) = \frac{1}{1 + t^2} \]

4. [12] Determine whether the following statements are true or false.

a) If \( X(f) \) is the Fourier transform of \( x(t) \), then \( |X(f)| \) is always an even function.

\[ \text{F} \]

b) The spectrum of a periodic signal is always discrete.

\[ \text{T} \]

c) For a continuous time signal, \( x(t) = 10sinc(10t) \) the Nyquist rate of sampling is 5Hz.

\[ \text{F} \]

d) For an even symmetric periodic signal, \( a_k \) s are equal to zero in the trigonometric Fourier series expansion, \( x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2\pi kf_0 t) + \sum_{k=1}^{\infty} b_k \sin(2\pi kf_0 t) \).

\[ \text{F} \]
Part B- Show all your work to get partial credit. Correct answers without complete work will not receive full credit.

1. [36] For the following system

\[ p(t) = \sum_{k=-\infty}^{\infty} \delta(t - 0.5k) \]
\[ X(f) = \text{tri}(f) \]
\[ H(f) = \begin{cases} 
1, & -2 \leq f \leq -1, \quad 1 \leq f \leq 2 \\
0, & \text{otherwise}
\end{cases} \]
\[ G(f) = \begin{cases} 
1, & -1 \leq f \leq 1 \\
0, & \text{otherwise}
\end{cases} \]

Sketch A(f), B(f), C(f) and Y(f). Show amplitudes and frequencies.
Extra Sheet for Question 1:

\[ A(f) = X(f) \ast 2 \sum_{k=-\infty}^{\infty} s(f-2k) = \sum_{k=-\infty}^{\infty} X(f-2k) \]

\[ B(f) \]

\[ C(f) \]

\[ Y(f) \]
Extra Sheet for Question 1:
2. [37] For the following periodic signal,
\[ x(t) = 2 - 4 \cos(8\pi t) + 10 \sin(12\pi t - \frac{\pi}{3}) \]

(a) [4] Find the fundamental frequency of the signal.

(b) [15] Find the exponential Fourier series coefficients.

Hint: You can first find the trigonometric Fourier series coefficients and then convert them to the exponential Fourier series.

\[ \sin(a - b) = \sin(a) \cos(b) - \sin(b) \cos(a) \]
\[ \cos(\frac{\pi}{3}) = \frac{1}{2}, \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2} \]

(c) [8] Find the power of the signal from the Fourier series coefficients.

(d) [10] When \( x(t) \) is used as an input to the following ideal highpass filter, \( H(f) \), find the steady-state output, \( y(t) \).

\[ |H(f)| = \begin{cases} 1, & |f| \geq 5 \\ 0, & \text{otherwise} \end{cases} \]
\[ \angle H(f) = \begin{cases} \frac{-f\pi}{12}, & |f| \geq 5 \\ 0, & \text{otherwise} \end{cases} \]

\[ f_1 = 4 \text{ Hz}, \quad f_2 = 6 \text{ Hz} \quad f_0 = \text{GCD}(4, 6) = 2 \text{ Hz}. \]

\[ x(t) = 2 - 4 \cos(8\pi t) + 10 \left[ \sin(12\pi t) \cos(\frac{\pi}{3}) - \cos(12\pi t) \sin(\frac{\pi}{3}) \right] \]

\[ x(t) = 2 - 4 \cos(8\pi t) + 5 \sin(12\pi t) - 5\sqrt{3} \cos(12\pi t) \]

\[ a_0 = 2 \]
\[ a_1 = 0 \]
\[ b_1 = 0 \]
\[ a_2 = -4 \]
\[ b_2 = 0 \]
\[ a_3 = -5\sqrt{3} \]
\[ b_3 = 5 \]

\[ x[0] = 2 \]
\[ x[1] = 0, \quad x[-1] = 0 \]
\[ x[2] = -2, \quad x[-2] = -2 \]
\[ x[3] = \frac{1}{2} (-5 \sqrt{3} + 5) \frac{1}{2} \]
\[ x[-3] = \frac{1}{2} \left( -5 \sqrt{3} + 5 \right) \]

(all other coefficients are zero).
Extra Sheet for Question 2:

\[ P = \sum_{k=\infty}^{\infty} |x_k|^2 \]
\[ = 4 + (2)(4) + 2 \left( \frac{75}{4} + \frac{25}{4} \right) \]
\[ = 12 + 50 \]
\[ = 62 \text{ W/} \]

\[ y(t) = 10 |H(6)| \sin(12\pi t - \frac{\pi}{3} + \angle H(6)) \]
\[ = 10 \sin(12\pi t - \frac{\pi}{3} - \frac{\pi}{2}) = 10 \sin(12\pi t - 150^\circ) \]

\[ \angle H(6) = \frac{-6 \pi}{12} = -\frac{\pi}{2} \]
Extra Sheet for Question 2:
### Table 9.2 Operational Properties of the Fourier Transform

<table>
<thead>
<tr>
<th>Property</th>
<th>( x(t) )</th>
<th>( X(f) )</th>
<th>( X(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Similarity</td>
<td>( x(t) )</td>
<td>( x(-f) )</td>
<td>( 2\pi x(-\omega) )</td>
</tr>
<tr>
<td>Time Scaling</td>
<td>( v(at) )</td>
<td>( \frac{1}{</td>
<td>a</td>
</tr>
<tr>
<td>Folding</td>
<td>( x(-t) )</td>
<td>( X(-f) )</td>
<td>( X(\omega) )</td>
</tr>
<tr>
<td>Time Shift</td>
<td>( x(t-a) )</td>
<td>( e^{-j2\pi ft}X(f) )</td>
<td>( e^{-j2\pi \omega t}X(\omega) )</td>
</tr>
<tr>
<td>Frequency Shift</td>
<td>( e^{j2\pi ft}x(t) )</td>
<td>( X(f-a) )</td>
<td>( X(\omega - 2\pi a) )</td>
</tr>
<tr>
<td>Convolution</td>
<td>( x(t) \ast h(t) )</td>
<td>( X(f)H(f) )</td>
<td>( X(\omega)H(\omega) )</td>
</tr>
<tr>
<td>Multiplication</td>
<td>( x(t)h(t) )</td>
<td>( X(f) \ast H(f) )</td>
<td>( \frac{1}{2\pi} X(\omega) \ast H(\omega) )</td>
</tr>
<tr>
<td>Modulation</td>
<td>( x(t)\cos(2\pi at) )</td>
<td>( 0.5[X(\omega + a) + X(\omega - a)] )</td>
<td>( 0.5[X(\omega + 2\alpha a) + X(\omega - 2\alpha a)] )</td>
</tr>
<tr>
<td>Derivative</td>
<td>( x'(t) )</td>
<td>( j2\pi fX(f) )</td>
<td>( j\omega X(\omega) )</td>
</tr>
<tr>
<td>Times-t</td>
<td>( -2\pi itz(t) )</td>
<td>( X'(f) )</td>
<td>( 2\pi X'(\omega) )</td>
</tr>
<tr>
<td>Integration</td>
<td>( \int_{-\infty}^{\infty} x(t)dt )</td>
<td>( \frac{1}{j2\pi f} X(f) + 0.5X(0)\delta(f) )</td>
<td>( \frac{1}{j2\pi \omega} X(\omega) + \pi X(0)\delta(\omega) )</td>
</tr>
<tr>
<td>Conjugation</td>
<td>( x^*(t) )</td>
<td>( X(-f) )</td>
<td>( X(-\omega) )</td>
</tr>
<tr>
<td>Correlation</td>
<td>( x(t) \ast y(t) )</td>
<td>( X(f)Y(f) )</td>
<td>( X(\omega)Y(\omega) )</td>
</tr>
<tr>
<td>Autocorrelation</td>
<td>( x(t) \ast x(t) )</td>
<td>( X(f)X(f) =</td>
<td>X(f)</td>
</tr>
</tbody>
</table>

### Fourier Transform Theorems

- **Central ordinates**
  \( x(0) = \int_{-\infty}^{\infty} X(f)df = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)d\omega \)

- **Parseval’s theorem**
  \( \int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} |X(f)|^2df = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2d\omega \)

- **Plancherel’s theorem**
  \( \int_{-\infty}^{\infty} x(t)y(t)dt = \int_{-\infty}^{\infty} X(f)Y(f)df = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)Y(\omega)d\omega \)

### Table 11.2 Operational Properties of the Laplace Transform

**Note:** \( x(t) \) is to be regarded as the causal signal \( x(t)u(t) \).

<table>
<thead>
<tr>
<th>Entry</th>
<th>Property</th>
<th>( x(t) )</th>
<th>( X(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Superposition</td>
<td>( ax_1(t) + bx_2(t) )</td>
<td>( aX_1(s) + bX_2(s) )</td>
</tr>
<tr>
<td>2</td>
<td>Times-exp</td>
<td>( e^{-st}x(t) )</td>
<td>( X(s) )</td>
</tr>
<tr>
<td>3</td>
<td>Times-cos</td>
<td>( \cos(\alpha x(t)) )</td>
<td>( 0.5[X(s+\alpha) + X(s-\alpha)] )</td>
</tr>
<tr>
<td>4</td>
<td>Times-sin</td>
<td>( \sin(\alpha x(t)) )</td>
<td>( j0.5[X(s+\alpha) - X(s-\alpha)] )</td>
</tr>
<tr>
<td>5</td>
<td>Time Scaling</td>
<td>( x(at) ), ( a &gt; 0 )</td>
<td>( \frac{1}{a}X\left(\frac{s}{a}\right) )</td>
</tr>
<tr>
<td>6</td>
<td>Time Shift</td>
<td>( x(t-a)u(t-a) ), ( a &gt; 0 )</td>
<td>( e^{-as}X(s) )</td>
</tr>
<tr>
<td>7</td>
<td>Times-t</td>
<td>( tz(t) )</td>
<td>( -\frac{dX(s)}{ds} )</td>
</tr>
<tr>
<td>8</td>
<td>Derivative</td>
<td>( x'(t) )</td>
<td>( sX(s) - x(0-) )</td>
</tr>
<tr>
<td>9</td>
<td>Integral</td>
<td>( \int_{0}^{\infty} x(t)dt )</td>
<td>( \frac{x(s)}{s} )</td>
</tr>
<tr>
<td>10</td>
<td>Convolution</td>
<td>( x(t) \ast h(t) )</td>
<td>( X(\omega) )</td>
</tr>
<tr>
<td>11</td>
<td>Switched periodic, ( x_0(t) = \text{first period} )</td>
<td>( x_0(t)u(t) )</td>
<td>( X_0(s) ), ( T = \text{time period of } x(t) )</td>
</tr>
</tbody>
</table>

### Table 11.3 A Short Table of Laplace Transforms

<table>
<thead>
<tr>
<th>Entry</th>
<th>Property</th>
<th>( x(t) )</th>
<th>( X(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \delta(t) )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( u(t) )</td>
<td>( \frac{1}{s} )</td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td>3</td>
<td>( t \ast u(t) )</td>
<td>( \frac{1}{s^2} )</td>
<td>( \frac{1}{s^3} )</td>
</tr>
<tr>
<td>4</td>
<td>( \delta(t)u(t) )</td>
<td>( \frac{1}{s} )</td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td>5</td>
<td>( e^{-at}u(t) )</td>
<td>( \frac{1}{s+a} )</td>
<td>( \frac{1}{(s+a)^2} )</td>
</tr>
<tr>
<td>6</td>
<td>( te^{-at}u(t) )</td>
<td>( \frac{1}{(s+a)^2} )</td>
<td>( \frac{1}{(s+a)^3} )</td>
</tr>
<tr>
<td>7</td>
<td>( e^{-at} )</td>
<td>( \frac{1}{s+a} )</td>
<td>( \frac{1}{(s+a)^2} )</td>
</tr>
<tr>
<td>8</td>
<td>( \sin(at)u(t) )</td>
<td>( \frac{1}{s^2+a^2} )</td>
<td>( \frac{1}{s^2+a^2} )</td>
</tr>
<tr>
<td>9</td>
<td>( \cos(at)u(t) )</td>
<td>( \frac{s}{s^2+a^2} )</td>
<td>( \frac{s}{s^2+a^2} )</td>
</tr>
</tbody>
</table>

### Laplace Transform Theorems

<table>
<thead>
<tr>
<th>Entry</th>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>Initial value</td>
<td>( x(0+) = \lim_{t \to 0^+} x(t) ) (if ( x(s) ) is strictly proper)</td>
</tr>
<tr>
<td>16</td>
<td>Final value</td>
<td>( x(\infty) = \lim_{t \to \infty} x(t) ) (if poles of ( x(s) ) lie in LHP)</td>
</tr>
</tbody>
</table>