Filter Design

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For $h(n)$ with length $N$ we have $\sum_n h(n) = \sqrt{2}$ and $\sum_n h(n)h(n-2k) = \delta(k)$ which result in $\frac{N}{2} + 1$ equations for $N$ unknowns.

Therefore, there are $N - (\frac{N}{2} + 1) = \frac{N}{2} - 1$ degrees of freedom.

Example: For $N = 2$, there are zero degrees of freedom. The only length 2 filter that will result in a valid orthonormal wavelet family is Haar filter.

How do we use the remaining degrees of freedom? Regularity and vanishing moments.
Vanishing Moments

- Most applications of wavelet bases require the representation of signals with few non-zero wavelet coefficients.
- This depends on the regularity of the function, the number of vanishing moments of $\psi$ and the size of its support.
- $\psi$ has $K$ vanishing moments if $\int t^k \psi(t) dt = 0$ for $0 \leq k < K$. This means that $\psi$ is orthogonal to any polynomial of degree $K - 1$.
- If $f$ is regular (continuous) and $\psi$ has enough vanishing moments then the wavelet coefficients $| \langle f, \psi_{j,n} \rangle |$ are small at fine scales.
- This is due to the fact that it is possible to represent a continuous function locally using Taylor’s series expansion.
Theorem: Vanishing Moments

Let $\psi$ and $\phi$ be a wavelet and a scaling function that generate an orthogonal basis. The four following statements are equivalent:

1. The wavelet $\psi$ has $K$ vanishing moments.
2. $\Psi(\omega)$ and its first $K - 1$ derivatives are zero at $\omega = 0$.
3. $H(\omega)$ and its first $K - 1$ derivatives are zero at $\omega = \pi$.
4. For any $0 \leq k < K$, $q_k(t) = \sum_n n^k \phi(t - n)$ is a polynomial of degree $k$.

Proof: The $k$th order derivative $\psi^k(\omega)$ is the Fourier transform of $(-jt)^k \psi(t)$. Hence $\psi^k(0) = \int (-jt)^k \psi(t) dt$. If $\psi(t)$ has $K$ vanishing moments then $\psi(\omega)$ will have its first $K - 1$ derivatives equal to zero at $\omega = 0$. 
From the wavelet equation,
\[ \sqrt{2} \psi(2\omega) = e^{-j\omega} H^*(\omega + \pi) \Phi(\omega). \]
Therefore, 2 is equivalent to 3.

4 implies 1. Since \( \psi \) is orthogonal to \( \phi(t - n) \), it is also orthogonal to \( q_k, 0 \leq k < K \). This family of polynomials is a basis for the space of polynomials of degree at most \( K - 1 \). Hence, \( \psi \) is orthogonal to any polynomial of degree \( K - 1 \) and in particular to \( t^k \), \( \psi \) has \( K \) vanishing moments.
Support of $\psi$

If $f$ has an isolated singularity at $t_0$ and if $t_0$ is inside the support of $\psi_{j,n}(t)$, then $\langle f, \psi_{j,n} \rangle$ may have a large amplitude. If $\psi$ has a compact support of size $K$, at each scale $j$ there are $K$ wavelets $\psi_{j,n}$ whose support includes $t_0$. To minimize the number of high amplitude coefficients we must reduce the support size of $\psi$. We need to relate the support size of $h$ to the support of $\phi$ and $\psi$. 
Compact Support

The scaling function $\phi$ has a compact support iff $h$ has a compact support and their support are equal. If the support of $h$ and $\phi$ is $[N_1, N_2]$ then the support of $\psi$ is $[(N_1 - N_2 + 1)/2, (N_2 - N_1 + 1)/2]$.

Proof: If $h$ has support $[N_1, N_2]$ and $\phi$ has compact support $[K_1, K_2]$, the support of $\phi(t/2)$ is $[2K_1, 2K_2]$.

$$\frac{1}{\sqrt{2}} \phi(t/2) = \sum h(n)\phi(t - n)$$  \hspace{1cm} (1)

Therefore, $N_1 = K_1, N_2 = K_2$.

For the wavelet function,

$$\frac{1}{\sqrt{2}} \psi(t/2) = \sum (-1)^n h(1 - n)\phi(t - n)$$  \hspace{1cm} (2)

Therefore, $\psi(t)$ has support $\left[\frac{N_1 - N_2 + 1}{2}, \frac{N_2 - N_1 + 1}{2}\right]$.
Support vs. Moments

- If $\psi$ has $K$ vanishing moments then its support is at least of size $2K - 1$. Daubechies wavelets are optimal in the sense that they have a minimum size support for a given number of vanishing moments.

- There is a tradeoff between vanishing moments and support. If $f$ has few nonsingularities and is very regular between singularities, choose a wavelet with many vanishing moments. If there are too many singularities, decrease the support of $\psi$, lower the number of vanishing moments.

- If $h[n]$ is a regular filter then the corresponding scaling function will be smooth. A scaling filter is $K$-regular if its $z$ transform has $K$ zeros at $z = e^{j\pi} = -1$. Any unitary scaling filter has at least one zero at $z = -1$ since $H(\pi) = 0$. 
Daubechies Compactly Supported Wavelets

- Daubechies wavelets have a support of minimum size for any given number of $K$ vanishing moments.
- From the proposition, we know that wavelets with compact support are computed with FIR conjugate mirror filters, $h$.
- To ensure that $\psi$ has $K$ vanishing moments, $H(\omega)$ must have a zero of order $K$ at $\omega = \pi$. Therefore,

$$H(\omega) = \sqrt{2} \left( \frac{1 + e^{-j\omega}}{2} \right)^K Q(e^{-j\omega}) \quad (3)$$

- Theorem: A real conjugate mirror filter, $h$, such that $H(\omega)$ has $K$ zeros at $\omega = \pi$ has at least $2K$ nonzero coefficients. Daubechies filters have $2K$ nonzero coefficients.
Since $h[n]$ is real, $|H(\omega)|^2$ is an even function and can thus be written as a polynomial in $\cos(\omega)$. Hence $|R(e^{-j\omega})|^2$ is a polynomial in $\cos(\omega)$ that we can write as a polynomial $P(\sin^2(\omega/2))$, $|H(\omega)|^2 = 2(\cos^2\omega)^2K P(\sin^2\omega)$. The quadrature filter condition is equivalent to $(1 - y)^K P(y) + y^K P(1 - y) = 1$ (let $y = \sin^2(\omega/2)$). To minimize the number of nonzero terms of $H(\omega)$, we must find the solution $P(y) \geq 0$ of minimum degree, which is obtained with the Bezout theorem on polynomials.
Bezout Theorem

Let $Q_1(y)$ and $Q_2(y)$ be two polynomials of degrees $n_1$ and $n_2$ with no common zeros. There exist two unique polynomials $P_1(y)$ and $P_2(y)$ of degrees $n_2 - 1$ and $n_1 - 1$ such that $P_1(y)Q_1(y) + P_2(y)Q_2(y) = 1$.

In our case, $Q_1(y) = (1 - y)^K$, $n_1 = K$, $Q_2(y) = y^K$, $n_2 = K$. Therefore, $P_1(y), P_2(y)$ have degrees $K - 1$. If we solve this equation, we can verify the $P_2(y) = P_1(1 - y) = P(1 - y)$ and

$P(y) = \sum_{k=0}^{K-1} \binom{K - 1 + k}{k} y^k$. 
We need to find $H(\omega)$. We know $|Q(\omega)|^2 = P(\sin^2(\omega/2))$.

$$Q(\omega)Q^*(\omega) = Q(\omega)Q(-\omega) = P\left(\frac{2 - e^{-j\omega} - e^{-j\omega}}{4}\right) = R(\omega))$$

(4)

In z domain, $Q(z) = q(0) \prod_{k=0}^{m}(1 - a_k e^{-j\omega})$. Therefore, $Q(z)Q(z^{-1}) = q^2(0) \prod_{k=0}^{m}(1 - a_k z)(1 - a_k z^{-1}) = R(z) = P\left(\frac{2-z-z^{-1}}{4}\right)$.

To find $Q(z)$, find roots of $R(z)$. Since $R(z)$ has real coefficients if $c_k$ is a root, $c_k^*$ is a root. Since it’s a function of both $z$ and $z^{-1}$, if $c_k$ is a root, $1/c_k$ and $1/c_k^*$ are also roots.

To design $Q(z)$, choose each root among a pair such that it’s inside the unit circle.

Since $P(z)$ has degree $K - 1$, $h[n]$ has length $K + K - 1 + 1 = 2K$. 
Symmlets

- Daubechies wavelets are very asymmetric because they are constructed by selecting the minimum phase roots. Filters with minimum phase have their energy concentrated near the starting point of their support.
- To obtain a symmetric or antisymmetric wavelet, $h$ must be symmetric, $H(\omega)$ has a linear complex phase. Haar is the only real compactly supported QMF that has a linear phase.
- Symmlets are obtained by optimizing the choice of the square root to obtain almost linear phase. The resulting filters will still have $K$ vanishing moments, but will be more symmetric.
- We can design complex QMF filters with a compact support and linear phase.
Coiflets (Coifman)

- Wavelets that have $K$ vanishing moments and a minimum size support, but whose scaling functions also satisfy $\int \phi(t) dt = 1$ and $\int t^k \phi(t) dt = 0$, $1 \leq k < K$.

- The scaling functions also have vanishing support.

- The minimum length of the corresponding wavelet will be $3K - 1$ (instead of $2K - 1$).

- Scaling function is more symmetric and provides better approximation.

- The minimum length Coiflet is length 6.
Cascade Algorithm: Iterative algorithm to generate successive approximations to $\phi(t)$. The iterations are defined by

$$
\phi^{k+1}(t) = \sum_{n=0}^{N-1} h(n) \sqrt{2} \phi^k(2t - n)
$$

(5)

An initial $\phi^0$ needs to be chosen. If the algorithm converges to a fixed point, then that fixed point is the scaling function.

Similarly, one can compute the wavelet function using the filter $h_1$.

These iterative algorithms can also be implemented in the frequency domain:

$$
\Phi^{k+1}(\omega) = \frac{1}{\sqrt{2}} H(\frac{\omega}{2}) \Phi^k(\frac{\omega}{2})
$$

(6)