Introduction to Signal Spaces

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Outline

1 Motivation
2 Vector Space
3 Inner Product Space
4 Normed Vector Space
5 $L_2$ and $l_2$ Spaces
6 Basis
7 Fourier Series
Our goal in this course is to expand any signal $x$ from some space $S$ in terms of elementary signals $\{\phi_i\}$ such as
$$x = \sum_i \alpha_i \phi_i.$$  

The set $\{\phi_i\}$ should be complete for $S$.

The expansion coefficients, $\alpha_i$ should be sparse, explain the signal with as few coefficients as possible. Attractive for signal processing applications like compression.

There should be a fast algorithm to compute $\alpha_i$ from the signal.

$\{\phi_i\}$ should be a meaningful set of functions, e.g. sinusoids.
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A linear vector space $X$ is a collection of vectors (over the complex or real field) together with two operations; vector addition and scalar multiplication, which for all $x, y \in X$ and $\alpha \in \mathbb{C}/\mathbb{R}$ satisfy the following:

1. **Commutativity:** $x + y = y + x$ for all $x, y \in X$.
2. **Associativity:** $x + (y + z) = (x + y) + z$.
3. **Distributivity:** $\alpha(x + y) = \alpha x + \alpha y$.
4. **Additive Identity:** There is a zero vector such that $x + 0 = x$.
5. **Additive Inverse:** For each $x \in X$ there is a unique vector $-x$ such that $x + (-x) = 0$.
6. **Multiplicative Identity:** $1x = x$. 
Examples

- Set of real numbers ($\mathbb{R}$): It is a real vector space with addition defined in the usual way and multiplication by scalars defined as ordinary multiplication. The null vector is the real number zero.
- n-dimensional real or complex vector spaces: $\mathbb{R}^n, \mathbb{C}^n$.
- $L_p[0, T] = \{f(t) | \int |f(t)|^p dt < \infty\}, 1 \leq p < \infty$. Used for representing continuous time signals.
- $l_p = \{x[n] | \sum |x[n]|^p < \infty\}, 1 \leq p < \infty$. Used for discrete time signals.
- $C[0, T]$: The collection of all real-valued continuous functions on the interval $[0, T]$.
- The collection of functions that are piecewise constant between integers.
- $\mathbb{R}^+$ a vector space?
Subspace

- A subspace, $S$, of a vector space $V$ is a subset of $V$ that is closed under vector addition and scalar multiplication.

- Example: Is $X_1 = \{x \in C[0, T] : \int_0^T x(t)dt = 1\}$ a subspace of $C[0, T]$?

- Answer: No, if $x, y \in X_1$, then check whether $x + y \in X_1$.

$$\int_0^T x(t) + y(t)dt = \int_0^T x(t)dt + \int_0^T y(t)dt = 2 \neq 1 \quad (1)$$
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Inner Product

An inner product of two vectors $x, y \in V$ on a vector space is a complex-valued scalar assigned to the two vectors and satisfies the following properties:

1. $< x, y > = ( < y, x > )^*$.
2. $< \alpha x, y > = \alpha < x, y >$.
3. $< x + y, z > = < x, z > + < y, z >$.
4. $< x, x > \geq 0, < x, x > = 0$ iff $x = 0$.

An inner product space (pre-Hilbert space) is a linear vector space in which an inner product can be defined for all elements of the space and a norm is given by $\|x\| = ( < x, x > )^{1/2}$.

Cauchy-Schwarz Inequality: For all $x, y$ in an inner product space, $| < x, y > | \leq \|x\| \|y\|$. Equality holds if and only if $x = \lambda y$ or $y = 0$. 
A complete inner product space is called a Hilbert space. Completeness means that every sequence of vectors in the vector space converge to a vector in the vector space, e.g. the space of rational numbers is not complete.

A sequence of vectors \( \{x_n\} \) is called a Cauchy sequence if \( \|x_n - x_m\| \to 0 \) as \( n, m \to \infty \). If every Cauchy sequence in \( V \) converges to a vector in \( V \), then \( V \) is called complete.

Example: \( V = \mathbb{C}^N, \langle x, y \rangle = y^H x = \sum_{i=1}^{N} x_i y_i^* \).

Example: \( V = L_2[a, b], \langle x, y \rangle = \int_a^b x(t)y^*(t)dt \).

In this course, the space of signals we will consider will be a Hilbert space.
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A norm is a function \( (\| \cdot \| : V \to \mathbb{R}) \) such that the following properties hold:

1. \( \|x\| \geq 0 \) with equality iff \( x = 0 \).
2. \( \|\alpha x\| = |\alpha|\|x\| \).
3. Triangle Inequality: \( \|x + y\| \leq \|x\| + \|y\| \) with equality iff \( x = \alpha y \).

The norm measures the size of a vector. A vector space together with a norm function is called the normed vector space.

The notion of length (norm) gives meaning to the distance between two vectors in \( V \), \( d(x, y) = \|x - y\| \).
Normed Vector Space: Examples

- For example, $V = \mathbb{R}^N$, $\|x\| = \sqrt{\sum_{i=1}^{N} x_i^2}$.
- $V = l_p$, $\|x\| = (\sum_n x^p[n])^{1/p}$.
- The normed space $C[a, b]$ consists of continuous functions on the interval $[a, b]$ together with the norm $\|x\| = \max_{a \leq t \leq b} |x(t)|$.
- You can define different norms for the same vector space yielding different normed spaces.
- A norm can be defined directly from an inner product, though not every norm can be defined from an inner product.
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**L₂ Space**

- For an interval \( a \leq t \leq b \), the space \( L₂([a, b]) \) is the set of all square integrable functions defined on \( a \leq t \leq b \), i.e.
  \[
  L₂([a, b]) = \{ f : [a, b] \to \mathbb{C}; \int_a^b |f(t)|^2 \, dt < \infty \}.
  \]
- This corresponds to the space of all signals whose energy is finite.
- Functions that are discontinuous are allowed as members of this space.
- The space \( L₂[a, b] \) is infinite dimensional.
- If \( a = 0 \) and \( b = 1 \), then \( f(t) = 1/t \) is an example of a function that does not belong to \( L₂[0, 1] \).
- The inner product on \( L₂[a, b] \) is defined as
  \[
  \langle f, g \rangle = \int_a^b f(t)g^*(t) \, dt.
  \]
For many applications, the signal is already discrete. In such cases, we represent the signal as a sequence.

The space $l_2$ is the set of all sequences with $\sum_n |x_n|^2 < \infty$.

The inner product on this space is defined as $\langle x, y \rangle = \sum_n x_n y_n^*$.
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A necessary and sufficient condition for the set of vectors \( x_1, x_2, \ldots, x_n \) to be linearly independent is that the expression \( \sum_{k=1}^{n} \alpha_k x_k = 0 \) implies \( \alpha_k = 0 \) for all \( k = 1, 2, 3, \ldots, n \).

The span of set \( S \subset V \) is the subspace of \( V \) containing all linear combinations of vectors in \( S \). When \( S = \{ x_1, x_2, \ldots, x_N \} \), \( \text{span}(S) = \{ \sum_{i=1}^{N} \alpha_i x_i \} \).

A subset of linearly independent vectors \( \{ x_1, x_2, \ldots, x_N \} \) is called a basis for \( V \) when its span equals to \( V \). In this case, we say that the dimension of \( V \) is equal to \( N \). \( V \) is infinite-dimensional if it contains an infinite number of linearly independent vectors.

Any two bases for a finite-dimensional vector space contain the same number of elements.
Once you have a basis for a vector space you can express
any vector as a linear combination of the elements of this
set.

Example: For $\mathbb{R}^3$, the basis is $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

Example: The space of infinite sequences, such as $l_2$, is
spanned by the infinite set \( \{ \delta[n - k] \}_{k \in \mathbb{Z}} \). Since they are
linearly independent, the space is infinite-dimensional.

Example: The space of second order polynomials, $P_2$ is
spanned by \( \{ 1, x, x^2 \} \).

Note: In infinite dimensional spaces, the concept of basis
is a bit different since we have not defined infinite sums.
Orthogonality

- The vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be orthogonal if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
- The collection of vectors, $e_i, i = 1, \ldots, N$, is said to be orthogonal if each $e_i$ has unit length and $\langle e_i, e_j \rangle = \delta_{ij}$.
- Two subspaces $V_1$ and $V_2$ of $V$ are said to be orthogonal if each vector in $V_1$ is orthogonal to every vector in $V_2$. 
Orthogonal and Orthonormal Basis

A set of vectors \( \{ x_1, x_2, \ldots, \} \) form an orthonormal basis for \( V \) if they span \( V \) and \( < x_i, x_j > = \delta_{ij} \).

In this course, we will work with infinite dimensional function spaces such as \( L_2 \) and \( l_2 \).

In Hilbert space, orthonormal bases are more useful. Any orthonormal set in a Hilbert space can be extended to form an orthonormal basis.

Fourier Series Theorem: Let \( \{ x_n \} \) be an orthonormal set in a Hilbert space, \( H \). Then the following statements are equivalent:

1. \( \{ x_n \} \) is an orthonormal basis.
2. For any \( x \in H \), \( x = \sum_n < x, x_n > x_n \).
3. For any two vectors, \( < x, y > = \sum_n < x, x_n > < y, y_n > \).
4. For any \( x \), \( ||x||^2 = \sum_n | < x, x_n > |^2 \).
Orthogonal Projection Theorem

- Suppose $V_0$ is a subspace of an inner product space $V$. Suppose $\{e_1, e_2, \ldots, e_N\}$ is an orthonormal basis for $V_0$. Let $v \in V$ and define $v_0$ as the orthogonal projection of $v$ onto $V_0$, then $v_0 = \sum_{i=1}^{N} \langle v, e_i \rangle e_i$ and it is the vector closest to $v$ in the space $V_0$. Moreover, the error $v - v_0$ is orthogonal to $v_0$.

- Example: $V = \mathbb{R}^2$ and $V_0 = \mathbb{R}$, then $v_0 = \langle v, e_1 \rangle e_1$. In $\mathbb{R}^2$, the inner product is the dot product. Therefore, $v_0 = v \cos(\theta)$.

- In this course, we will consider orthogonal projections in infinite dimensional vector spaces. Wavelet expansion is equivalent to projecting a signal onto different subspaces.
Gram-Schmidt Orthogonalization

- Given any countable linearly independent set \( \{ y_n \} \) in an inner product space, it is always possible to construct an orthonormal set from it.
- We can construct an orthonormal set \( \{ x_n \} \) as follows:
  
  1. \[ x_1 = \frac{y_1}{\|y_1\|}. \]
  2. \[ x_{k+1} = \frac{y_{k+1} - \sum_{i=1}^{k} \langle y_{k+1}, x_i \rangle x_i}{\|y_{k+1} - \sum_{i=1}^{k} \langle y_{k+1}, x_i \rangle x_i\|}. \]
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There are many practical reasons for expanding a function as a trigonometric sum. If \( f(t) \) is a signal, then a decomposition of \( f \) into a trigonometric sum gives a description of its component frequencies.

The space of functions considered are periodic functions, \( L_2[-T/2, T/2] \) with the inner product

\[
\langle f, g \rangle = \frac{1}{T} \int_{-T/2}^{T/2} f(t) g^*(t) dt.
\]

The functions \( \{ e^{j\omega_0 kt} \} \) form an orthonormal basis.

By Fourier Theory, they span the space. Orthonormality can be proven as follows:

\[
\langle e^{j\omega_0 kt}, e^{j\omega_0 lt} \rangle = \frac{1}{T} \int_{-T/2}^{T/2} e^{j\omega_0 (k-l)t} dt = \frac{1}{T} \left[ \frac{e^{j(k-l)\omega_0 T/2}}{j(k-l)\omega_0} - \frac{e^{-j(k-l)\omega_0 T/2}}{j(k-l)\omega_0} \right]
\]
If we need to find the best approximation of $x(t)$ using $N$ terms in the summation, this is equivalent to finding the orthogonal projection of $x(t)$ onto a subspace spanned by 
\{1, e^{i\omega_0 t}, \ldots, e^{i\omega_0 Nt}\} and the expansion coefficients that will minimize the error are given by $\langle x(t), e^{i\omega_0 t} \rangle$.

Fourier transform is an extension for aperiodic signals and can be defined as $F(\omega) = \langle f(t), e^{i\omega t} \rangle$. 
Shannon’s Sampling Theorem

- If \( f(t) \) is continuous and bandlimited to \( \omega_m \), then \( f(t) \) is uniquely defined by its samples taken at the sampling frequency of \( 2\omega_m \) (the minimum sampling frequency, Nyquist rate), with \( T = \pi / \omega_m \).

\[
f(t) = \sum_{n=-\infty}^{\infty} f(nT) \text{sinc}_T(t - nT)
\]

\[
\text{sinc}_T(t) = \frac{\sin(\pi t / T)}{\pi t / T}
\]  

(3)

- This implies that the space of continuous and bandlimited functions form a vector space (\( U_T \) functions whose Fourier transform is limited to \([-\pi / T, \pi / T]\)) and \( h(t - nT) = \text{sinc}_T(t - nT) \) form an orthogonal basis for this space.

- In this case, \( f(nT) = \frac{1}{T} < f(t), h(t - nT) > \).