ECE 802-603 Homework 2 Solutions

1. a)

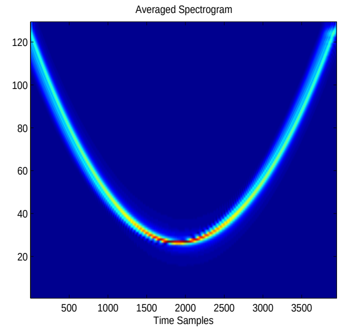

Figure 1: The average of the four spectrograms

b)The spectrogram with the window length 128 provides the best time and frequency resolution. The average spectrogram is better than N=32 and N=64 but also has extra ripple effects or uncertainty in frequency.

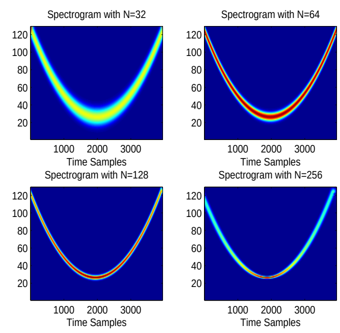

Figure 2: The average of the four spectrograms

2. Prove that the spectrogram can be written as convolution of Wigner distribution of the signal with Wigner distribution of the window.

$$P_{SP}(t,\omega) = \int\int W_s(t',\omega')W_h(t'-t,\omega-\omega')dt'd\omega' \tag{1}$$

To prove this equality, we'll use Moyal's formula.

$$P_{SP}(t,\omega) = \frac{1}{2\pi}|\int s(u)h(u-t)e^{-j\omega u}du|^2 \tag{2}$$

Using the formulation from the previous question, let $s_1(u) = s(u)$, and $s_2^*(u) = h(u-t)e^{-j\omega u}$. Therefore, using Moyal's formula, the spectrogram can be written as

$$P_{SP}(t,\omega) = \int\int W_{s_1}(u,\omega')W_{s_2}(u,\omega')dud\omega' \tag{3}$$

where $W_{s_1}(u,\omega') = W_s(u,\omega')$ since $s_1(u) = s(u)$. $W_{s_2}(u,\omega')$ is the Wigner distribution of $h^*(u-t)e^{j\omega u}$ and can be expressed in terms of Wigner distribution of the window.

$$\begin{aligned} W_{s_2}(u,\omega') &= \int h(u-t-\tau/2)h^*(u-t+\tau/2)e^{j\omega(u+\tau/2)}e^{-j\omega(u-\tau/2)}e^{-j\omega'\tau}d\tau \\ &= \int h(u-t-\tau/2)h^*(u-t+\tau/2)e^{j(\omega-\omega')\tau}d\tau \\ &= W_h^*(u-t,\omega-\omega') \\ &= W_h(u-t,\omega-\omega') \end{aligned} \tag{4}$$

Therefore,

$$P_{SP}(t,\omega) = \int\int W_s(u,\omega')W_h(u-t,\omega-\omega')dud\omega' \tag{5}$$

3. For $s(t) = A_1e^{j\omega_1 t} + A_2e^{j\omega_2 t}$,the instantaneous frequency can be found as follows. First we need to compute Wigner distribution of the signal. From the results developed in class

$$W(t,\omega) = A_1^2\delta(\omega-\omega_1) + A_2^2\delta(\omega-\omega_2) + 2A_1A_2\delta(\omega - \frac{\omega_1+\omega_2}{2})\cos((\omega_2-\omega_1)t) \tag{6}$$

The instantaneous frequency is defined as:

$$<\omega>_t = \frac{\int \omega W(t,\omega)d\omega}{|s(t)|^2} \tag{7}$$

where $|s(t)|^2 = A_1^2+A_2^2+2A_1A_2\cos((\omega_2-\omega_1)t)$. Therefore,the instantaneous frequency is:

$$<\omega>_t = \frac{A_1^2\omega_1 + A_2^2\omega_2 + 2A_1A_2\cos((\omega_2-\omega_1)t)\frac{\omega_1+\omega_2}{2}}{A_1^2+A_2^2+2A_1A_2\cos((\omega_2-\omega_1)t)} \tag{8}$$

For the particular signal given in this question, $A_1 = 0.4$,$A_2 = 1$,$\omega_1 = 10$, and $\omega_2 = 20$. In this case, the instantaneous frequency is:

$$\frac{21.6+12\cos(10t)}{1.16+0.8\cos(10t)} \tag{9}$$

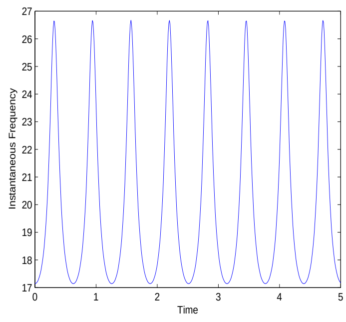

Figure 3: The instantaneous frequency versus time

This result does not match our intuition since the instantaneous frequency is outside of the range of frequencies that exist in this signal (10-20).
b)The Wigner distribution for this signal is from part (a):

$$W(t,\omega) = 0.16\delta(\omega - 10) + \delta(\omega - 20) + 0.8\delta(\omega - 15)\cos(10t) \tag{10}$$

c)function fr=instfr(tfd);
$[N, M]$=size(tfd);
vec=[0:1/(N-1):1]';
for k=1:M;
fr(k)=sum(vec.*tfd(:,k))/sum(tfd(:,k));
end
d) The actual instantaneous frequency can be obtained by extracting the ridges. However, in this example due to the cross-terms the closeness of the two frequency components, and the difference in the amplitudes of the two complex exponentials one of the frequency components is not exactly recovered.

4. a),b)The Wigner distribution and the Choi-Williams distribution for the given crossing chirps signal are:
c)We choose $\sigma = 1$ as the best distribution, a good compromise between cross terms and the auto terms. The initial frequency for the first chirp is 0.01 and the end frequency is 0.3. For the second chirp signal, the initial frequency is 0.5 and the end frequency is 0.
d)The ambiguity function is defined as $A(\theta, \tau) = \int s(u + \frac{\tau}{2})s^*(u - \frac{\tau}{2})e^{j\theta u}du$. For the given signal:

$$A(\theta, \tau) = \int e^{j\frac{\omega_0}{2}(u+\frac{\tau}{2})^2} e^{-j\frac{\omega_0}{2}(u-\frac{\tau}{2})^2} e^{j\theta u} du$$

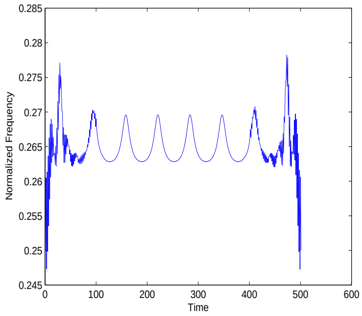

Figure 4: The instantaneous frequency from Wigner distribution versus time

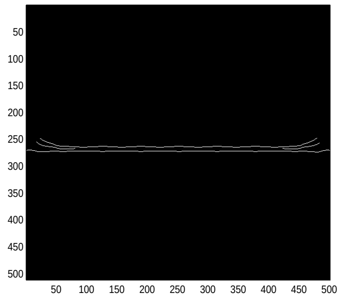

Figure 5: The instantaneous frequency from Wigner distribution versus time

$$
\begin{aligned}
&= \int e^{j\omega_0 u\tau} e^{j\theta u} du \\
&= 2\pi\delta(\theta + \omega_0\tau)
\end{aligned}
\tag{11}
$$

For the sum of two chirp signals the ambiguity function corresponding to the auto-terms will be $A(\gamma, \tau) = 2\pi\delta(\theta + \omega_1\tau) + 2\pi\delta(\theta + \omega_1\tau)$. Therefore, we can use a kernel that will filter out everything but the auto-terms as $\phi(\theta, \tau) = \delta(\theta + \omega_1\tau) + \delta(\theta + \omega_1\tau)$. e,f)

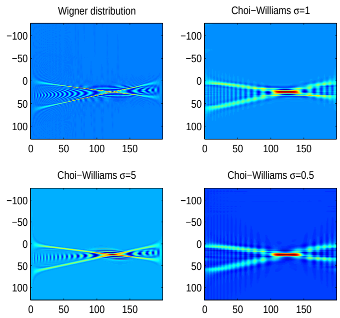

Figure 6: The Wigner and Choi-Williams distributions for the two crossing chirps

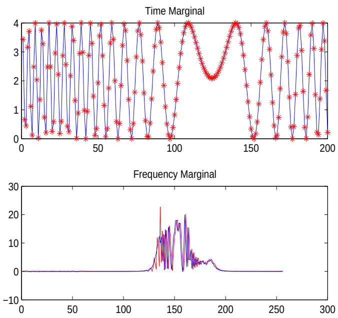

Figure 7: The time and the frequency marginals

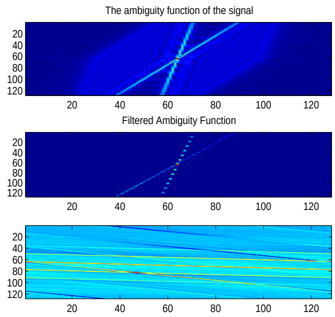

Figure 8: The sum of two chirp signals in the ambiguity domain and the time-frequency domain after filtering