ECE 802-603 Homework 1 Solutions

1. a) It is known that the symmetric (even) part of a signal can be obtained as:
$$f_s(x) = \frac{f(x) + f(-x)}{2} \tag{1}$$
Similarly, the antisymmetric (odd) part of a signal can be written as:
$$f_a(x) = \frac{f(x) - f(-x)}{2} \tag{2}$$
Therefore, $f(x) = f_s(x) + f_a(x)$.
b) For the space S, $\{\frac{cos(nx)}{\sqrt{2\pi}}\}$ form an orthonormal basis. Similarly, for the space A, $\{\frac{sin(nx)}{\sqrt{2\pi}}\}$ form an orthonormal basis. It is straightforward to show that both of these sets are orhonormal since $\frac{1}{2\pi}\int_{-\pi}^{\pi} cos(nx)cos(mx)dx = \delta_{nm}$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} sin(nx)sin(mx)dx = \delta_{nm}$. In order to establish that they form bases for the given spaces, we need to show that they span the space. By Fourier series theory, we can show that any function, $f(x) \in L_2[-\pi, \pi]$ can be written as a combination of $\{\frac{cos(nx)}{\sqrt{2\pi}}\}$, $\{\frac{sin(nx)}{\sqrt{2\pi}}\}$. For $f(x) \in S$, the expansion coefficients for $\{\frac{sin(nx)}{\sqrt{2\pi}}\}$ are equal to zero. Therefore, S is spanned by $\{\frac{cos(nx)}{\sqrt{2\pi}}\}$. A similar argument can be made for the subspace, A.
c)Projection onto V_1 is $v_1 = \frac{1}{2\pi}(\int_{-\pi}^{\pi} x^2 dx) + (\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} x^2 sin(x)dx)\frac{sin(x)}{\sqrt{\pi}} + (\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} x^2 cos(x)dx)\frac{cos(x)}{\sqrt{\pi}} = \frac{\pi^2}{3} - 4cos(x)$. Similarly, the orthogonal projection to V_2 can be found as $v_2 = v_1 + (\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} x^2 sin(2x)dx)\frac{sin(2x)}{\sqrt{\pi}} + (\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} x^2 cos(2x)dx)\frac{cos(2x)}{\sqrt{\pi}} = \frac{\pi^2}{3} - 4cos(x) + cos(2x)$. A similar calculation yields the projection onto V_3 as $v_3 = \frac{\pi^2}{3} - 4cos(x) + cos(2x) - \frac{4}{9}cos(3x)$.These projections are plotted in Figure 1.
d) The inner products are 2.0439, 1.4252 and 1.0805, respectively for the different subspaces. The reason why the inner products are not exactly equal to zero is that a continuous time inner product is approximated by a discrete summation in MATLAB.

2. a) a) We need to show that $\{1, t, t^2, \ldots\}$ is a linearly independent set. We will first show that any subset of these vectors are linearly independent. Therefore, $\sum_{i=0}^{N} \alpha_i t^i = 0$ for all t iff $\alpha_i = 0$. Writing the Nth

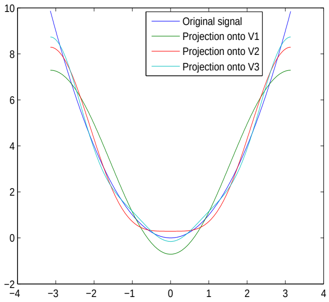

Figure 1: The signal and its projections

derivatives and computing α_is iteratively yield:

$$\begin{aligned}
\alpha_N(N)(N-1)(N-2)\ldots &= 0 \\
\alpha_N &= 0 \\
\alpha_{N-1}(N-1)(N-2)\ldots 1 + \alpha_N(N)(N-1)\ldots t &= 0 \\
\alpha_{N-1} &= 0 \qquad (3)
\end{aligned}$$

b) Gram-Schmidt procedure can be computed with the following MATLAB algorithm:

```
function z=gramschmidt(x,N,M);
for i=1:N;
s(i,:)=x(i,:);
end;
e(1)=s(1,:)*conj(s(1,:).');
phi(1,:)=s(1,:)/sqrt(e(1));
for i=2:N;
th(i,:)=zeros(1,M);
for r=i-1:-1:1;
th(i,:)=th(i,:)+(s(i,:)*conj(phi(r,:).'))*phi(r,:);
end;
```

```
th(i,:)=s(i,:)-th(i,:);
e(i)=th(i,:)*conj(th(i,:).');
phi(i,:)=th(i,:)/sqrt(e(i));
end;
z=phi(1:N,:);
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b)For the polynomial set $\{1, t, t^2, t^3, ...\}$, the corresponding orthonormal set is known as the Legendre polynomials. The first five functions in the orthonormal set is shown in Figure 2.

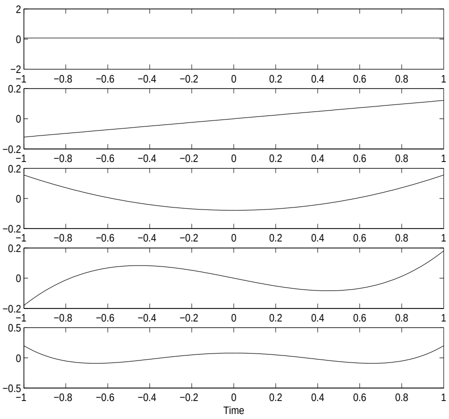

Figure 2: The first five orthonormal polynomials

3. a) Since $\{x_1, x_2, \ldots\}$ is an orthonormal basis, y_1 and y_2 can be written as $y_1 = \sum_i \alpha_i x_i$ and $y_2 = \sum_j \beta_j x_j$. Using these representations for y_1 and y_2, we can write the inner product as:

$$
\begin{aligned}
< y_1, y_2 > &= < \sum_i \alpha_i x_i, \sum_j \beta_j x_j > \\
&= \sum_i \sum_j \alpha_i \beta_j^* < x_i, x_j > \\
&= \sum_i \sum_j \alpha_i \beta_j^* \delta_{ij} \\
&= \sum_i \alpha_i \beta_j^*
\end{aligned}
$$

$$= \sum_i < y_1, x_i >< y_2, x_i >^* \quad (4)$$

where we used the definition of orthonormal basis and the expansion coefficients to simplify the equality.
b)Using the result from part (a) and the fact that $\frac{1}{\sqrt{2\pi}}e^{j\omega t}$ is an orthonormal basis and $< f, \frac{1}{\sqrt{2\pi}}e^{j\omega t} >= F(\omega)$, we get Parseval's equality.
c)Using the equality proven in part (a),

$$\begin{aligned} < h_T(t-nT), h_T(t-pT) > &= \frac{1}{2\pi}\int T^2 1_{[-\pi/T,\pi/t]}(\omega)e^{-j(n-p)T\omega}d\omega \\ &= \frac{T^2}{2\pi}\int_{-\pi/T}^{\pi/T} e^{-j(n-p)T\omega}d\omega \\ &= T\delta(n-p) \end{aligned} \quad (5)$$

Therefore, $h_T(t-nT)$ are an orthogonal set of functions. By Shannon's sampling theorem, if $f(t)$ is bandlimited in $[-\pi/T, \pi/T]$, then

$$f(t) = \sum_n f(nT)\frac{sin(\pi(t-nT)/T)}{\pi(t-nT)/T}.$$

This shows that for any $f(t) \in U$,$f(t) = \sum_n a(n)h_T(t-nT)$. Therefore, $h_T(t-nT)$ form a basis for U.The coefficients in the expansion can be obtained through inner product of $f(t)$ with $h_T(t-nT)$ since $h_T(t-nT)$ form an orthogonal basis.Therefore,

$$\begin{aligned} < f(t), h_T(t-nT) > &= \frac{1}{2\pi}\int F(\omega)H(\omega)e^{jnT\omega}d\omega \\ &= \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} F(\omega)e^{jnT\omega}d\omega \\ &= Tf(nT). \end{aligned} \quad (6)$$

Therefore, $f(nT) = \frac{1}{T} < f(t), h_T(t-nT) >$.

4. a) The Fourier series coefficients will be significant for 2 rad/sec, 4 rad/sec, 38 rad/sec and 42 rad/sec. Note the signal does not have to be necessarily periodic to define Fourier series coefficients. As long as the signal is defined in a finite interval, we can consider that interval as one period.

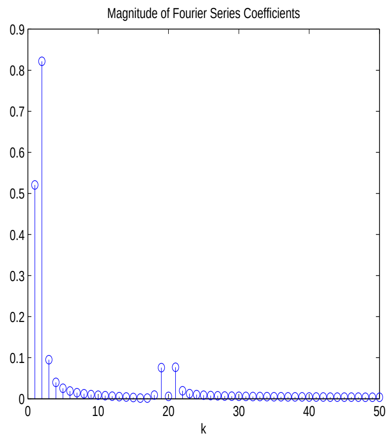

Figure 3: The magnitude of the Fourier series coefficients

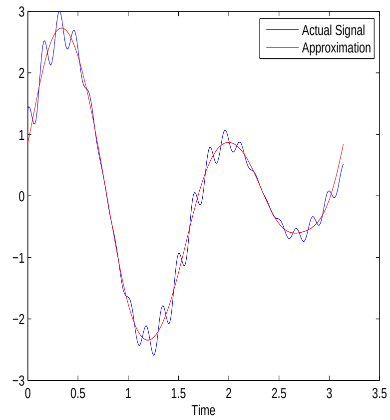

Figure 4: The magnitude of the Fourier series coefficients

b) The most significant coefficients are at $k = 1, 2, 19, 21$ corresponding to the frequencies identified in part (a).
c) x=0:.01:pi;

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f = exp(-(x.^2)/10).*(cos(2*x)+2*sin(4*x)+0.4*cos(2*x).*cos(40*x));
for k=1:50;
coef(k)=(1/length(x))*sum(f.*exp(-j*2*k*x));
end
dc=(1/length(x))*sum(f);
fpart=zeros(size(f))+dc;
for n=1:6;
fpart=fpart+coef(n)*exp(j*2*n*x)+conj(coef(n))*exp(-j*2*n*x);
end
```