ECE 802-603 Homework 1
Spring 2010
Due January 26, 2010

1. [25] Consider the space of square integrable functions on the interval $[-\pi,\pi]$, $L_2[-\pi,\pi]$, and the associated othonormal basis given by $\left\{1,\frac{\cos(nx)}{\sqrt{\pi}},\frac{\sin(nx)}{\sqrt{\pi}}\right\}, n=1,2,...$ Consider the following two subspaces, S: space of symmetric functions on $[-\pi,\pi]$ and A: space of antisymmetric functions on $[-\pi,\pi]$.

a) Show how any function $f(x)\in L_2[-\pi,\pi]$ can be written as $f(x)=f_s(x)+f_a(x)$, where $f_s(x)\in S$, $f_a(x)\in A$.
b) Give orthonormal bases for S and A.
c) Consider the subspaces $V_N \subset L_2[-\pi,\pi]$ spanned by $\left\{1,\frac{\sin(nx)}{\sqrt{\pi}},\frac{\cos(nx)}{\sqrt{\pi}};n=1,\ldots,N\right\}$. Find the orthogonal projection of the function $f(x)=x^2$ onto V_1,V_2,V_3. Plot these projections along with f using MATLAB.
d) Use MATLAB to verify that the projection error, i.e. $f(x)-f_{V_N}(x)$ is orthogonal to $f(x)$.

2. [25] Let $y_n(t)=t^n$ for $n=0,1,2,\ldots$ and $-1\le t\le 1$.

a) Show that $\{y_0,y_1,y_2,\ldots\}$ is a linearly independent set in $L_2[-1,1]$.
b) Write a MATLAB function that implements the Gram-Schmidt orthogonalization process. Apply it to $\{y_0,y_1,y_2,\ldots\}$ and plot the first four orthonormal vectors. Given a finite or countably infinite set of linearly independent vectors y_i, we can construct an orthonormal set x_i with the same span as follows:

- Start with $x_0=\frac{y_0}{\|y_0\|}$.
- Then, recursively set $x_1=\frac{y_1-<y_1,x_0>x_0}{\|y_1-<y_1,x_0>x_0\|}$
- Therefore, $x_k=\frac{y_k-v_k}{\|y_k-v_k\|}, v_k=\sum_{i=0}^{k-1}<y_k,x_i>x_i$

3. [25] Inner products are preserved by unitary transforms.

a) Prove the general Parseval's theorem, $<y_1,y_2>=\sum_i <y_1,x_i><y_2,x_i>^*$ for any y_1,y_2 in a vector space and an orthonormal basis, $\{x_1,x_2,x_3,\ldots\}$.

b) Using the result in part (a), show that

$$\int_{-\infty}^{\infty} f(t)h^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H^*(\omega)d\omega.$$

c) For $h_T(t) = \frac{\sin(\pi t / T)}{(\pi t / T)}$, prove that $\{h_T(t - nT)\}_{n \in Z}$ is an orthogonal basis of the space U of functions whose Fourier transforms are bandlimited in $[-\pi / T, \pi / T]$.

4. [25] Consider the function $f(x) = e^{-x^2/10}(\cos(2x) + 2\sin(4x) + 0.4\cos(2x)\cos(40x))$.

a) For what values of k would you expect the Fourier series coefficients $F[k]$ to be significant ? Why?

b) Compute the Fourier Series coefficients numerically using MATLAB for $k = 1,2,\ldots,50$ and verify your answer to part (a).

c) Plot the partial Fourier series summation through N=6 and compare with the plot of the original $f(x)$. Discuss your results.