Discussion #03 — January, 29, 2002

Problem **1** $y(t) = T[x,t],\ LTI,\ h(t) = T[\delta, t]$

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)}$$

Given the input $x(t)$ and the transfer function $h(t)$ as follows, find the output $y(t)$ of the system.

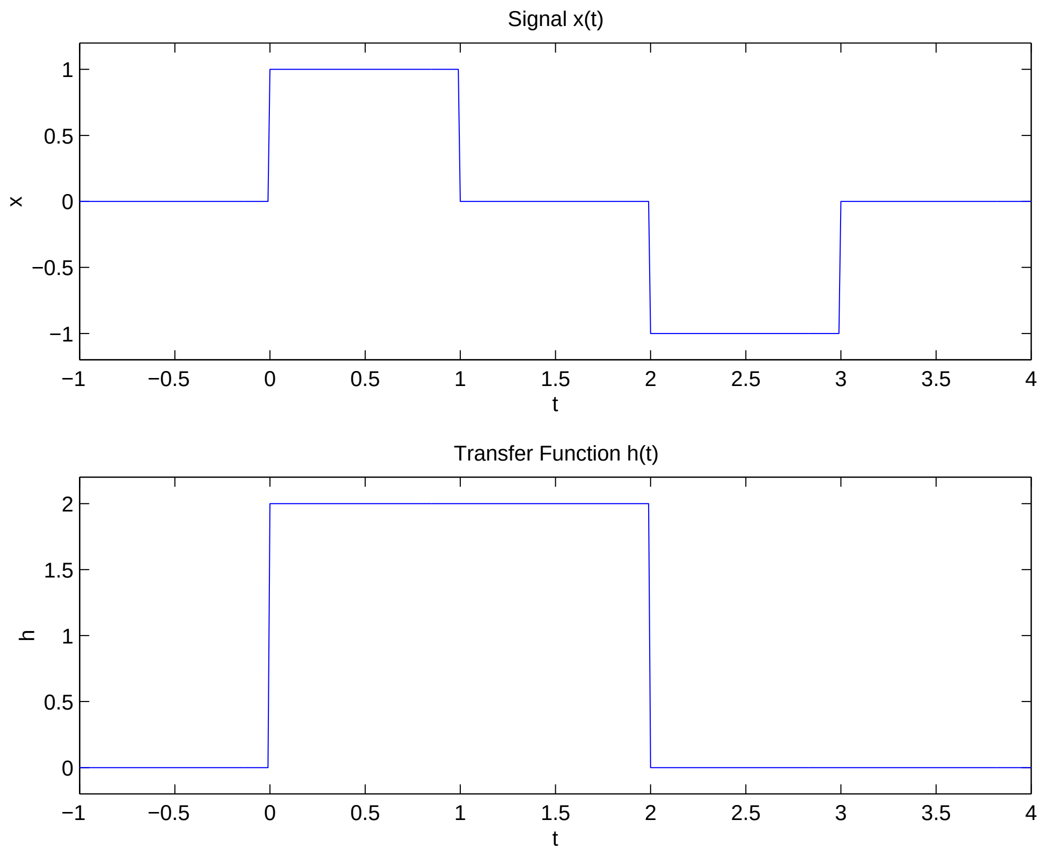

Figure 1: Input $x(t)$ and transfer function $h(t)$

Solution **General Procedure**

(1) Flip $h(\tau) \rightarrow h(-\tau)$

(2) Shift $h(\tau) \rightarrow h(-\tau - (-t)) = h(t - \tau)$

(3) Multiply $\boxed{h(t-\tau)}$ and $\boxed{x(\tau)}$

(4) Integrate $\boxed{h(t-\tau)x(\tau)}$ from $\boxed{-\infty \text{ to } \infty}$ over $\boxed{\tau}$

Let us apply the procedure to the given input and transfer function.

(1) Flip $h(\tau) \rightarrow h(-\tau)$

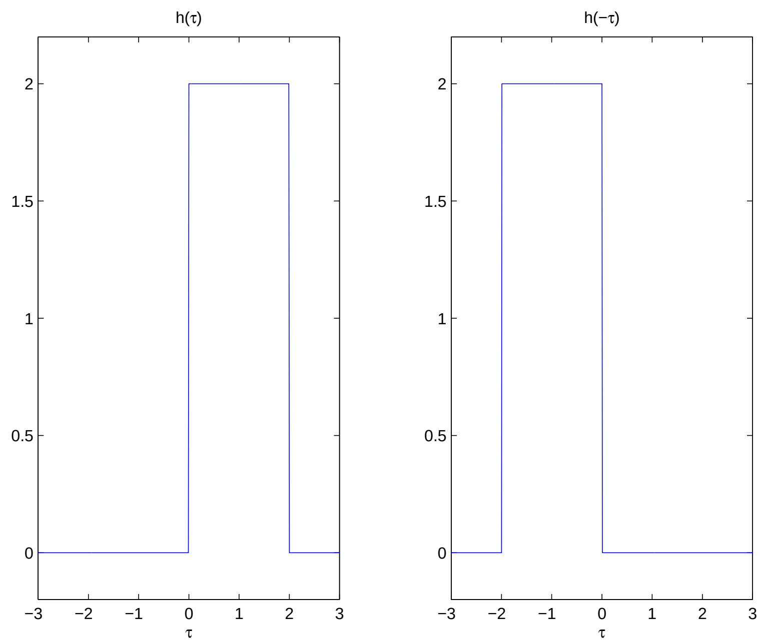

Figure 2: $h(\tau)$ and $h(-\tau)$

(2) Shift $h(\tau) \rightarrow h(-\tau - (-t)) = h(t - \tau)$

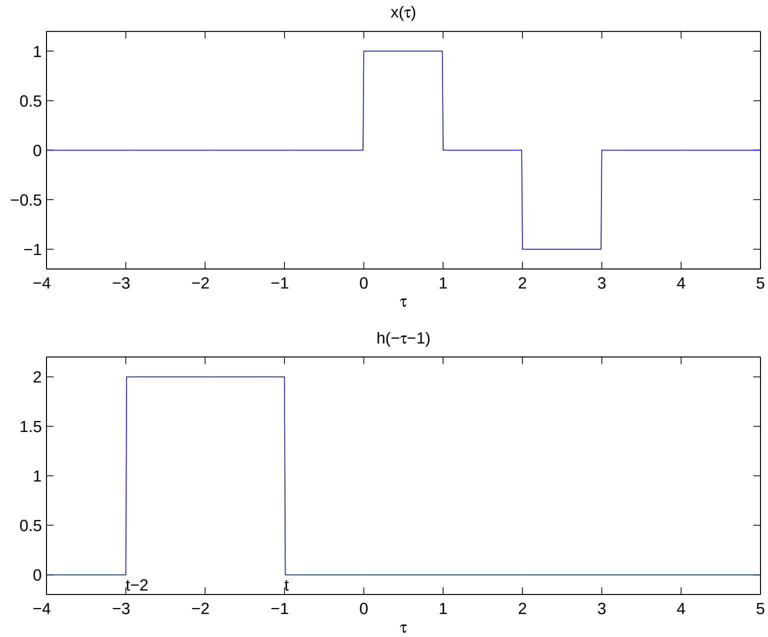

Figure 3: $h(-\tau + t)$, where $t < 0$

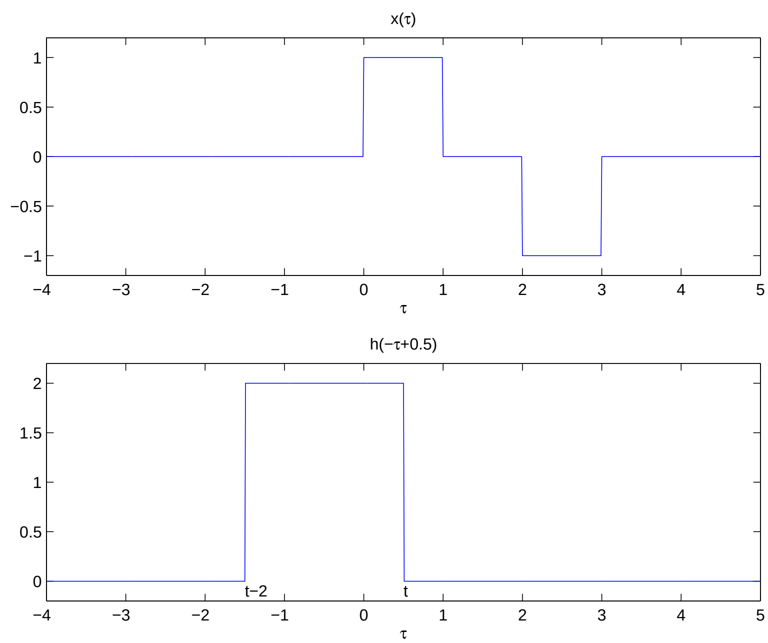

Figure 4: $h(-\tau + t)$, where $0 \leq t \leq 1$

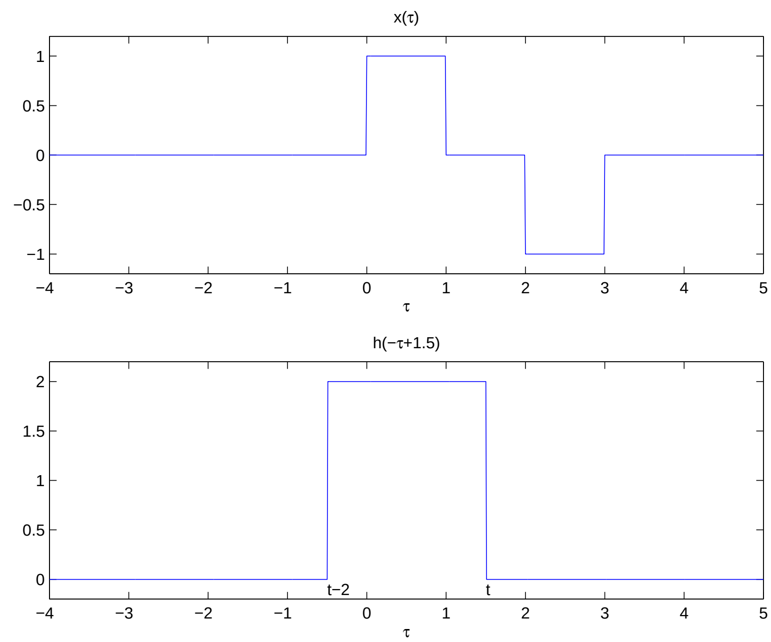

Figure 5: $h(-\tau + t)$, where $1 \leq t \leq 2$

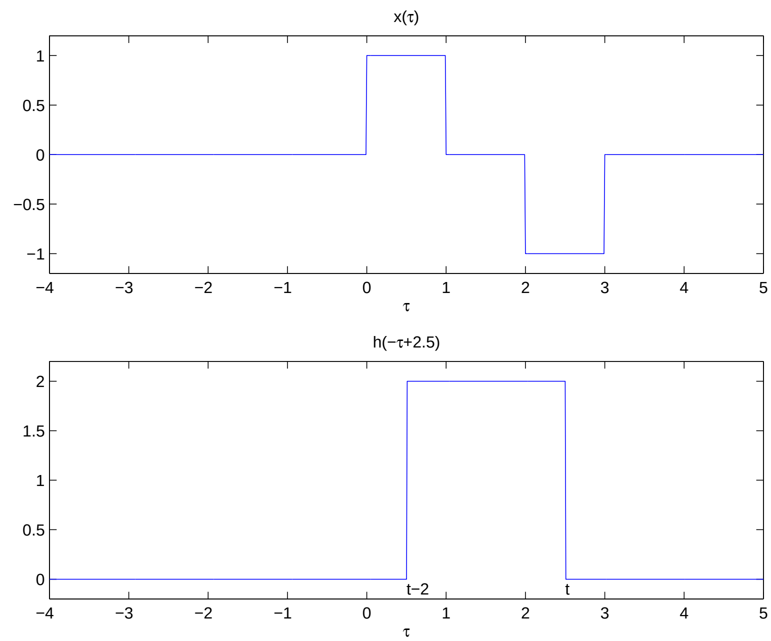

Figure 6: $h(-\tau + t)$, where $2 \leq t \leq 3$

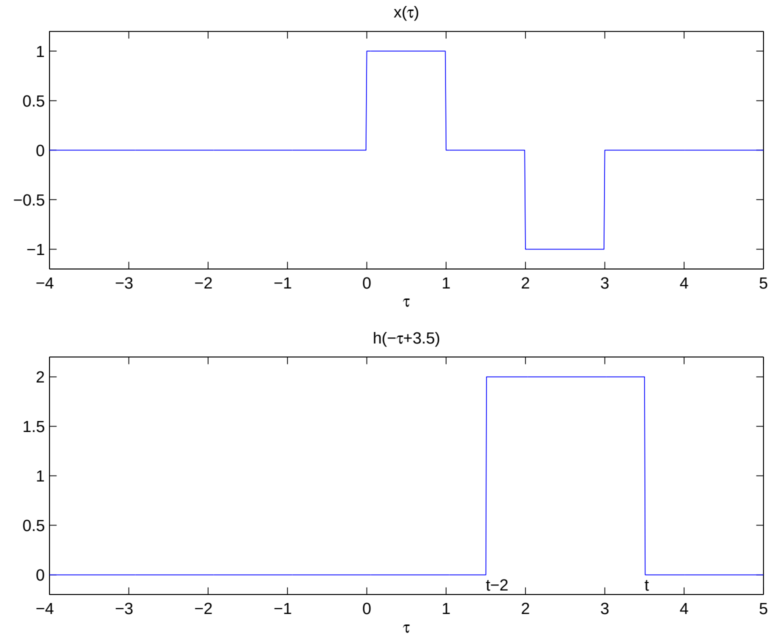

Figure 7: $h(-\tau + t)$, where $3 \leq t \leq 4$

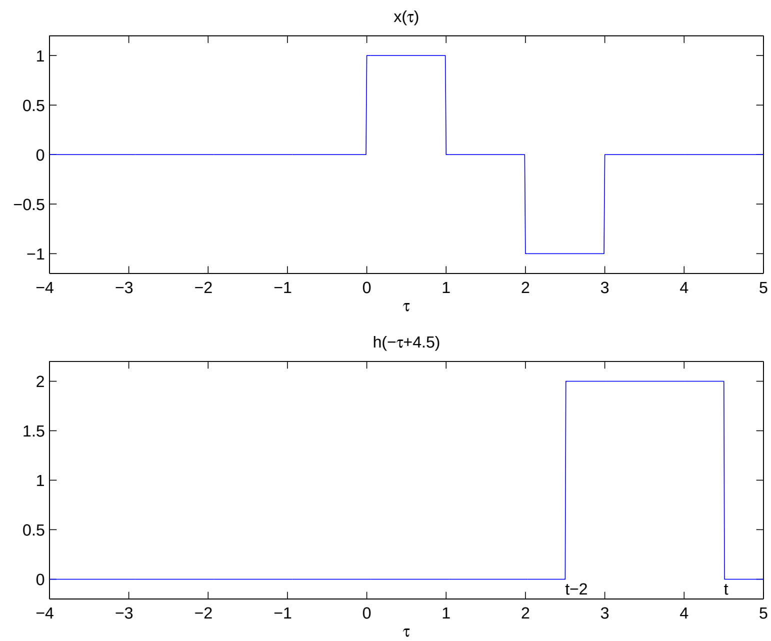

Figure 8: $h(-\tau + t)$, where $4 \leq t \leq 5$

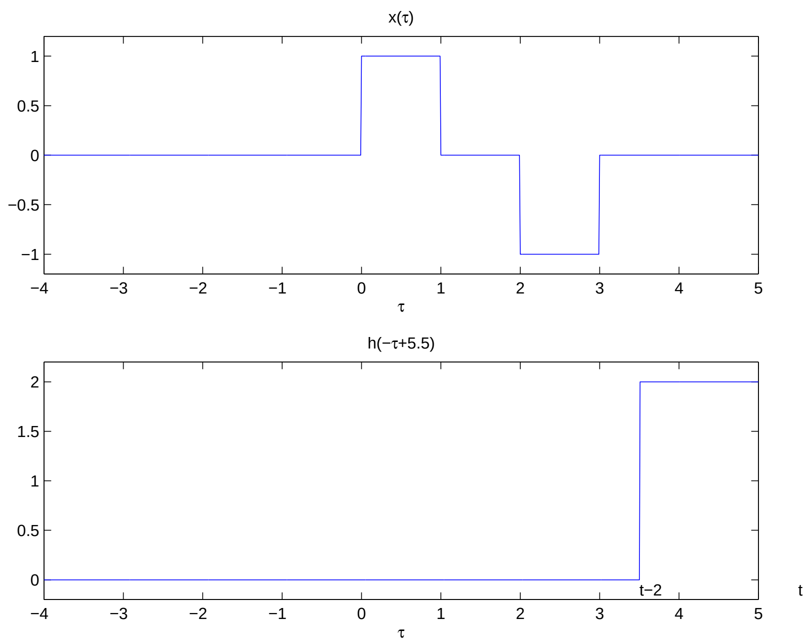

Figure 9: $h(-\tau + t)$, where $5 \leq t$

(3) Multiply $\boxed{h(t-\tau)}$ and $\boxed{x(\tau)}$

We are now going to multiply the two functions over the shaded areas. Shaded areas correspond to the nonzero values of the product of the two functions. But if there is none, this means that product is zero.

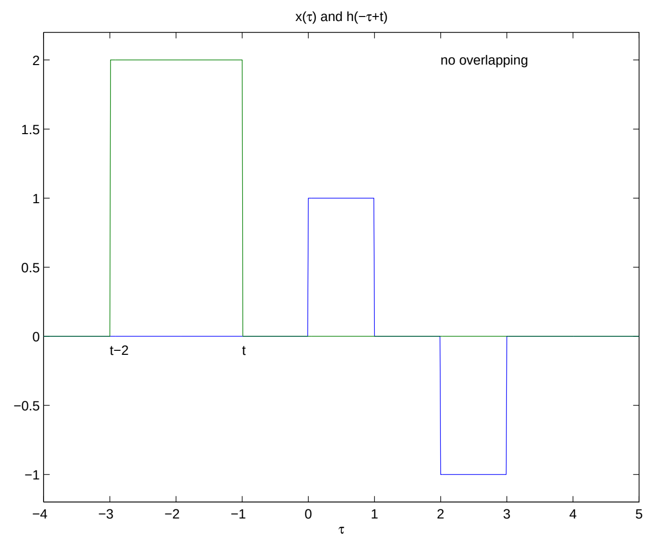

Figure 10: $h(-\tau+t)$, where $t<0$

(a) $t<0 \Rightarrow product = 0$

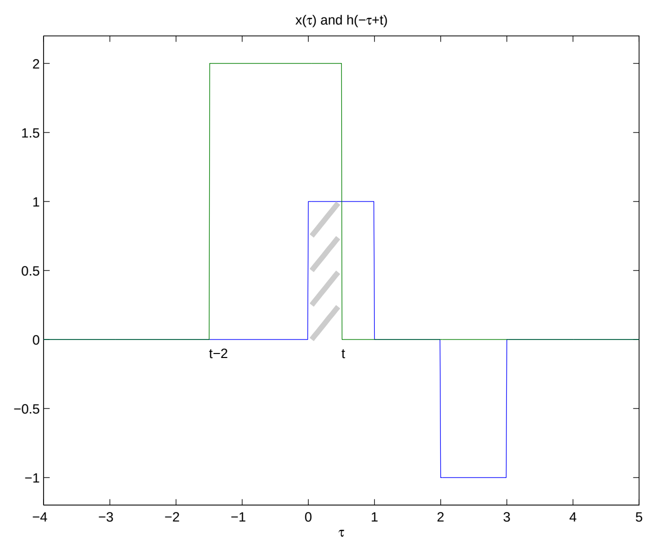

Figure 11: $h(-\tau+t)$, where $t<0$

(b) $0 \le t < 1 \Rightarrow h(t-\tau)x(\tau) = 2[u(\tau)-u(\tau-t)]$

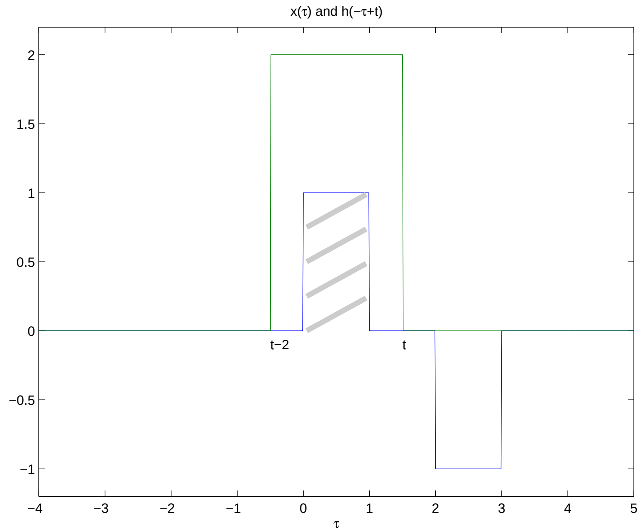

Figure 12: $h(-\tau + t)$, where $t < 0$

(c) $1 \leq t < 2 \Rightarrow h(t-\tau)x(\tau) = 2[u(\tau) - u(\tau - 1)]$

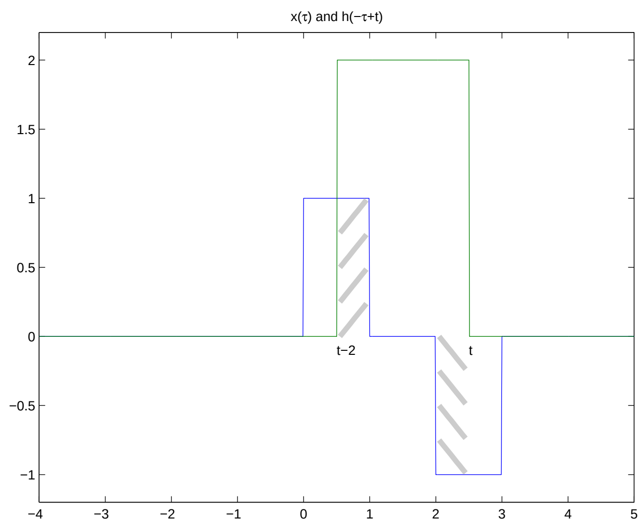

Figure 13: $h(-\tau + t)$, where $t < 0$

(d)

$$\begin{aligned} 0 \leq t-2 < 1 \Rightarrow 2 \leq t < 3 \quad & and \quad 2 \leq t < 3 \\ \Rightarrow h(t-\tau)x(\tau) \quad & = \quad 2[u(\tau - (t-2)) - u(\tau - 1)] - 2[u(\tau - 2) - u(\tau - t)] \end{aligned}$$

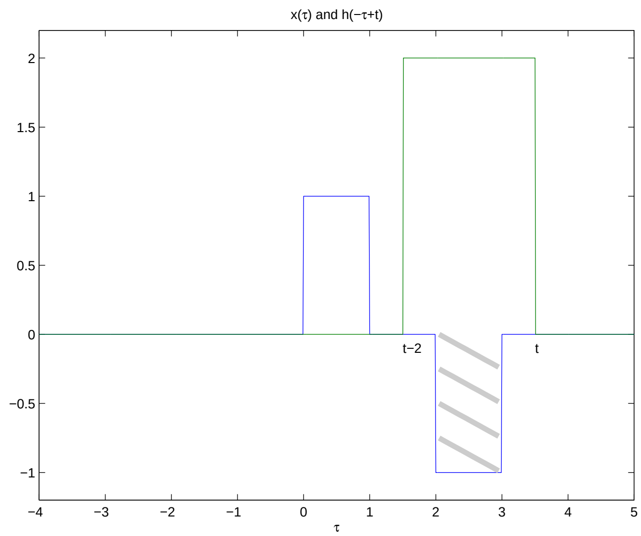

Figure 14: $h(-\tau + t)$, where $t < 0$

(e) $3 \leq t < 4 \Rightarrow h(t-\tau)x(\tau) = -2[u(\tau-2) - u(\tau-3)]$

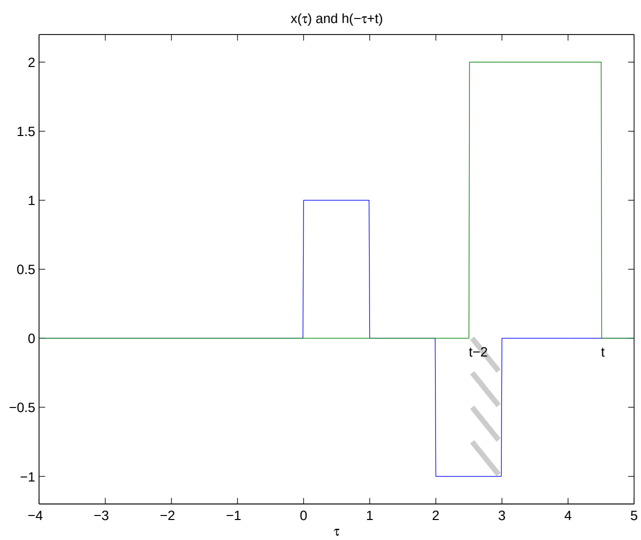

Figure 15: $h(-\tau + t)$, where $t < 0$

(f) $4 \leq t < 5 \Rightarrow h(t-\tau)x(\tau) = -2[u(\tau-(t-2) - u(\tau-3)]$

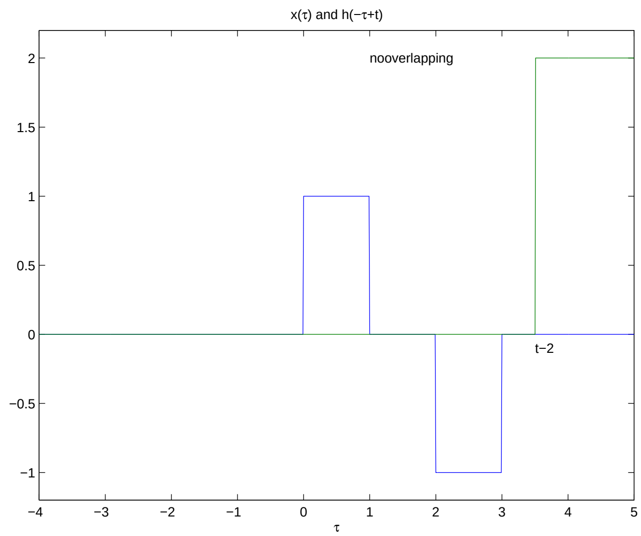

Figure 16: $h(-\tau + t)$, where $t < 0$

(g) $5 \leq t \Rightarrow h(t - \tau)x(\tau) = 0$

(4) Integrate

(a) $\boxed{t < 0}$

$$y(t) = 0$$

(b) $\boxed{0 \leq t < 1}$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{0}^{t} 2d\tau = \boxed{2t}$$

(c) $\boxed{1 \leq t < 2}$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{0}^{1} 2d\tau = \boxed{2}$$

(d) $\boxed{2 \leq t < 3}$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{t-2}^{1} 2d\tau + \int_{2}^{t} (-2)d\tau = 2(3-t) - 2(t-2) = \boxed{10 - 4t}$$

(e) $\boxed{3 \leq t < 4}$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{2}^{3} (-2)d\tau = \boxed{-2}$$

(f) $\boxed{4 \leq t < 5}$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{t-2}^{3} (-2)d\tau = -2(5-t) = \boxed{2(t-5)}$$

(g) $\boxed{5 \leq t}$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \boxed{0}$$

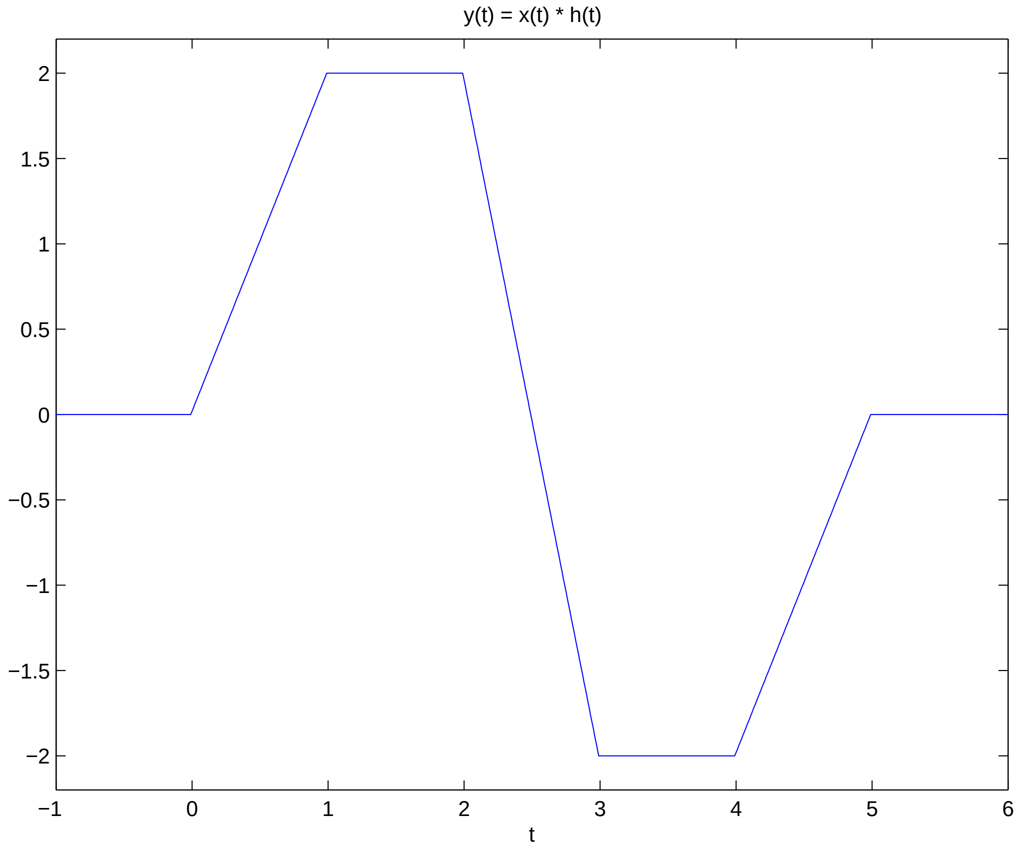

Figure 17: $y(t) = x(t) * h(t)$

***Solution* #2 Use the properties of convolution**

We can write the function $x(t)$ as a combination of two functions as follows. Let us define x_1 and x_2. We see that $\boxed{x(t) = x_1(t) + x_2(t)}$.

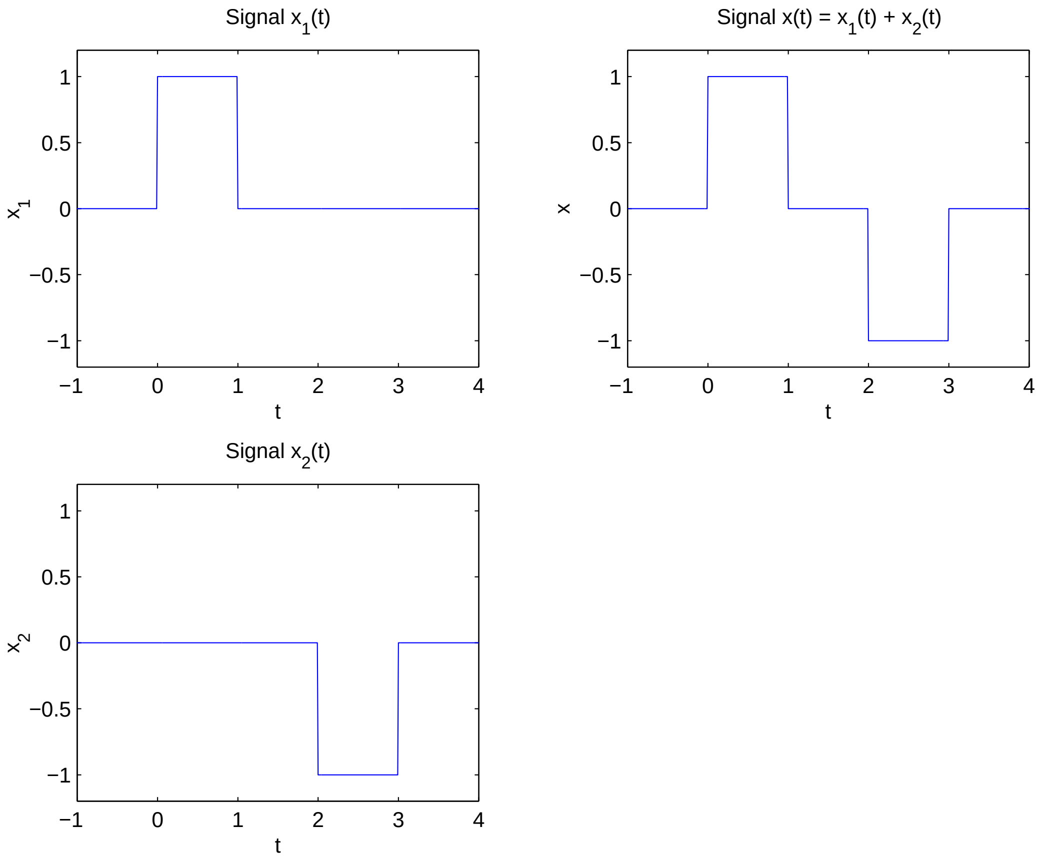

Figure 18: $x_1(t),\ x_2(t)$ and $x(t)$

Note:

$$x_2(t) = -x_1(t-2) = -S^2(x_1(t))$$

Let us now try to use our results above and the properties of convolution to find the output $y(t)$.

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) \\ y(t) &= h(t) * (x_1(t) + x_2(t)) \\ y(t) &= \underbrace{h(t) * x_1(t)}_{y_1(t)} + \underbrace{h(t) * x_2(t)}_{y_2(t)} \end{aligned}$$

$$\begin{aligned} y_1(t) &= h(t) * x_1(t) = T[x_1, t] \\ y_2(t) &= h(t) * x_2(t) = T[x_2, t] = T[-S^2(x_1), t] = -y_1(t-2) \\ y(t) &= y_1(t) - y_1(t-2) \end{aligned}$$

$$\begin{aligned} x_1(t) &= u(t) - u(t-1) \\ h(t) &= 2[u(t) - u(t-2)] \\ (h * x_1)(t) &= 2u(t) * u(t) - 2u(t) * u(t-2) - 2u(t-1) * u(t) + 2u(t-1) * u(t-2) \end{aligned}$$

Note: $u(t) * u(t) = tu(t)$

$$y_1(t) = (h * x_1)(t) = 2tu(t) - 2(t-2)u(t-2) - 2(t-1)u(t-1) + 2(t-3)u(t-3)$$

$$\Rightarrow y_2(t) = -y_1(t-2) = -2(t-2)u(t-2) + 2(t-4)u(t-4) + 2(t-3)u(t-3) - 2(t-5)u(t-5)$$

$$y(t) = y_1(t) + -y_1(t-2) = 2tu(t) - 4(t-2)u(t-2) - 2(t-1)u(t-1) + 4(t-3)u(t-3) \\ +2(t-4)u(t-4) - 2(t-5)u(t-5)$$

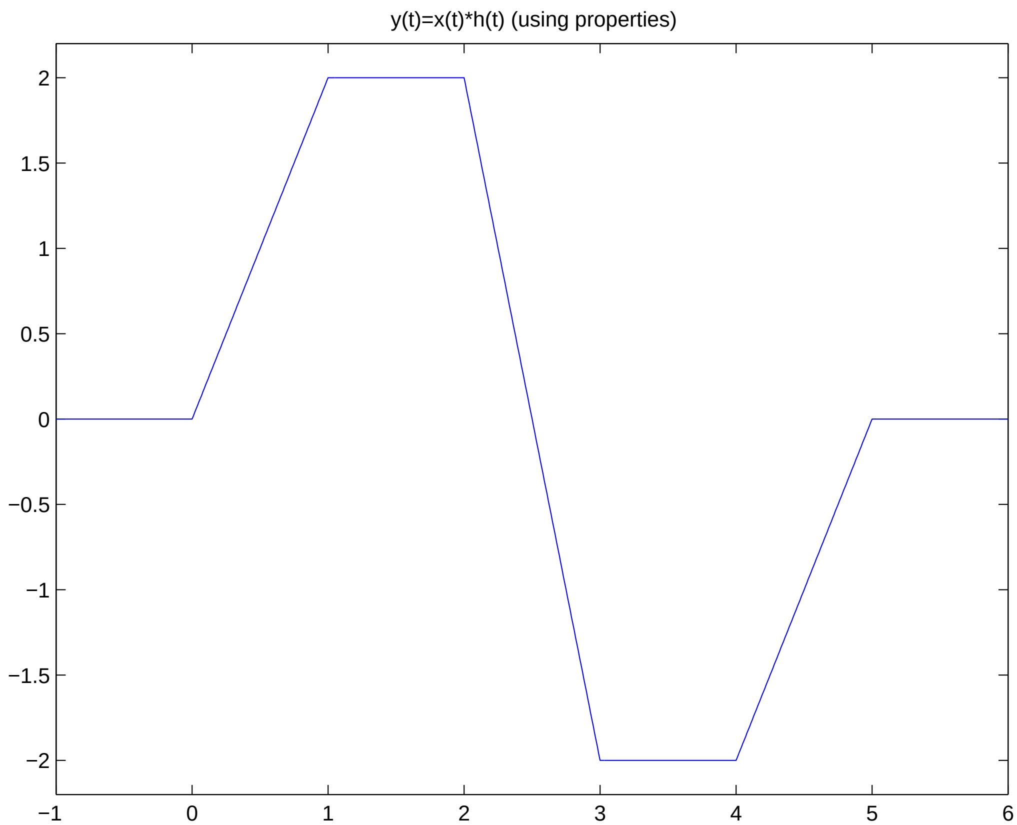

Figure 19: $y(t) = x(t) * h(t) that is find by using the properties$