

Density and Irregularity Dependence of Partial Recovery Codes

Shirish S. Karande and Hayder Radha

Department of Electrical & Computer Engineering / 2120 Engineering Building
Michigan State University
East Lansing, MI 48824 USA
{karandes, radha}@egr.msu.edu

Abstract—Classical linear block codes, such as Reed-Solomon (RS)-based codes, fail to recover any lost message symbols when the total losses exceed the redundant symbols. Under such adverse channel conditions, source and/or channel rate adaptation to incorporate additional redundancy might not be a viable option. In this paper, we explore a novel method of code adaptation which alters the degree distribution in order to achieve partial recovery of information when complete recovery is not possible. In particular, we change the degree distribution by adjusting the density and irregularity of the code. First, we illustrate that, while maintaining a constant rate, a partial recovery code can be optimized by density modification. Then, we focus on the Partial Reed-Solomon (PRS) codes, which are a family of RS-based codes that are capable of achieving different levels of partial recovery by adjustment to their order. We analyze the dependence of erasure recovery of these codes on density and regularity for a given number of losses. Finally, we present results and analysis which demonstrate that, for a given number of erasures, the PRS codes of order-1 render optimal (erasure-recovery) performance.

Keywords- FEC, Reed-Solomon, Density, Multimedia,

I. INTRODUCTION

With the advent of real-time services over data (wired/wireless) networks, Forward Error Correction (FEC) to facilitate delivery of delay-sensitive media has received tremendous research attention. Some recent studies (see for example [2], [3]) have outlined the time varying characteristics of contemporary networks (channels). Design of adaptive FEC schemes has, therefore, emerged as a vital research area.

Any block code is characterized by three parameters: 1) block-length (N); 2) dimension (K); and 3) coding rate (R). Modifications of a code can be realized by varying any two of the above parameters while retaining a constant value for the third. Most of the adaptive FEC schemes modify a code using one (or a combination) of the following schemes - augmenting, expurgating, extending, puncturing, lengthening, shortening [11]. The resulting (modified) code is more efficient and effective with respect to a certain (pre-defined) performance measure, e.g., higher decoding efficiency, lower complexity etc.

As the channel conditions deteriorate (i.e., the number of losses increase), most of the contemporary FEC schemes, in order to maximize information recovery, increase the

redundancy thereby altering the channel-coding rate. However, and as elaborated by [9], such coding rate modifications are severely constrained by bandwidth limitations. Rate control at the source encoder has been proposed as a potential solution to mitigate the above problem [9]. Nevertheless, in most applications source encoding is not done in real-time and the original uncompressed data is not necessarily available for renewed source encoding. Furthermore, performing source transcoding to lower the source rate is usually a high-complexity process making it infeasible for senders/servers that manage a large number of compressed streams. Consequently, there is a need for adaptive rate-constrained schemes which, while maintaining a constant coding rate, improve error recovery during adverse channel conditions.

Due to their generic nature and flexibility, graph codes have received a lot of research attention from the coding community. For example, [7] provides a graph code for erasure channels which demonstrates very low algorithmic complexity while retaining good decoding efficiency for low coding rates. Since linear block codes can be represented as graph codes, we believe that the application of graph coding concepts to the design of adaptive FEC requires further investigation. Hence, and in view of the above motivation, in this paper we analyze two parameters which have played a significant role in the design of graph codes, namely, the density and the degree distribution. For short binary codes, we provide extensive simulation results which depict that the erasure recovery of a linear block code is density dependent.

We show the existence of codes which can be employed to recover partial information in a scenario where the number of losses exceed the respective redundancy, i.e., complete recovery is not possible. Such codes, which we refer to as partial recovery codes ([4], [5]), are the primary focus of this work. In particular, we identify codes which maximize partial recovery for a given number of losses. Here, it is noteworthy that, due to their enhanced error resilience features ([1], [2]), emerging real-time applications can tolerate a certain level of losses in the multimedia content. Partial recovery of information can be used in conjunction with these error-resilience features to ensure a minimum multimedia quality even under severe (impaired) channel conditions.

We employ Low Density Parity Check (LDPC) codes as an example of non-trivial binary codes which are capable of partial recovery. The partial recovery performance of these

LDPC codes exhibits significant improvement potential. We, therefore, focus our attention solely to another family of partial recovery codes, namely Partial Reed Solomon (PRS) codes [4].

Throughout this paper, we particularly emphasize the impact of densities and regularities on constant-rate PRS codes. Within this context of constant coding rate, first we demonstrate that the error recovery performance of PRS codes is indeed sensitive to density and regularity variations. Furthermore, we illustrate that these parameters (i.e., density and regularity) can be varied without rate modification. More specifically, we demonstrate that these parameters can be modified as a function of the total number of losses.

The rest of this paper is organized as follows: Section II provides simulation results outlining that the erasure recovery of a linear block code is density dependent. We briefly illustrate the existence of partial recovery for non-trivial binary codes. In Section III we provide background on the design of PRS codes and present throughput analysis for a general PRS code. In Section IV and V we show that the recovery performance of PRS codes is density and regularity dependent. Finally, Section VI identifies the optimal PRS codes as a function of the number of losses¹. Section VII summarizes key conclusions of this paper.

II. DENSITY DEPENDENCE OF AN AVERAGE BINARY CODE

In this section, we present results of exhaustive simulations carried out for very short block length codes. Although the codes experimented with have short block lengths, the inferences derived from this example can be applied to codes with larger block lengths. A decoding algorithm proposed in [7] is used for all the following simulations. For a given (N, K) , the number of possible codes or code graphs are $2^{K \times (N-K)}$. It should be noted that for a systematic binary code the number of entries in the non-trivial part of the generator matrix are $K \times (N-K)$. Thus the number of possible generator matrices is $2^{K \times (N-K)}$. Naturally, some of the trivial codes, although a part of the code ensemble, do not provide efficient codes. This set of all possible codes can be divided into smaller subsets depending on the density of their graphs. Thus for a (N, K) code, the density can be varied by varying the number of edges in the code graph from 0 to $K \times (N-K)$. The erasure recovery performance for a given number of erasures is averaged over each subset. We assume that all erasure patterns of equal weight are equiprobable. Since a short block length is employed, we exhaustively generate every (possible) erasure pattern. The erasure recovery performance is, hence, averaged over all possible permutations of erasures with a constant weight (L) and over all possible codes of a given density.

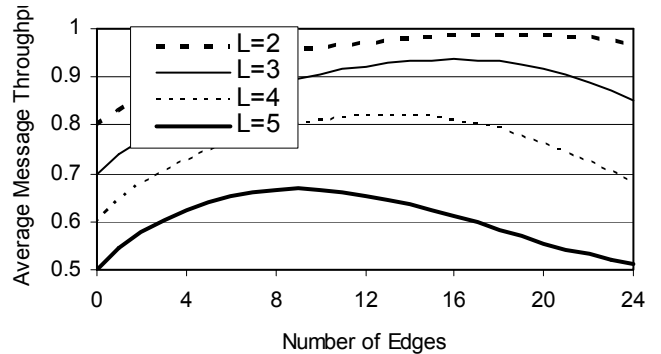


Figure 1. Message throughput of an average binary code.

Due to the averaging nature of our simulation setup, Figure 1 shows the message throughput of an *average* binary code for $N = 10, K = 6$. This figure shows the message throughput as a function of the number of edges for $L = 2$ to 5. The maximum number of edges in the code graph is equal to $K \times (N - K)$. It can be observed that the optimal density is a function of the number of losses. In addition, for a given value of L , a decrease in coding rate improves the performance of the optimal density code. For a given block-length and coding rate, as the number of losses increase the value of the optimal density decreases.

A. Partial recovery and density dependence of LDPC codes

In order to explore non-trivial binary codes we experimented with the $(3, 6)$ Gallager code with $R = 0.66$ and $N = 36000$ bits. The performance of such a code was observed for a BEC channel which is operating near and above channel capacity. It was observed that the current LDPC codes are capable of facilitating partial recovery of information while operating above channel capacity. However, the amount of partial recovery rendered by LDPC codes is, nevertheless, much lower than the redundancy and is, therefore, quite insignificant. Similarly, it was noted that LDPC codes do not provide perfect (complete) recovery even while operating below channel capacity. This example was used to highlight the fact that design of efficient partial recovery codes is a rather challenging task. Although the decoding efficiency of LDPC codes is inferior to RS codes, they can be employed by low-complexity applications. This highlights the efficacy of density adaptation for partial recovery while operating below channel capacity. However, based on our performance metric (i.e., decoding efficiency), an RS-based code can be much more effective. Henceforth, all subsequent analysis focuses on RS-based codes. On account of the brevity of space we are unable to present details about partial recovery performance of LDPC codes.

III. PARTIAL REED-SOLOMON CODES

For a given real-time pair constraint (N, K) , we denote a general PRS code of order s by $(N, K, \Lambda_s)_q$. Here q

¹ The focus of this paper is to identify effective partial recovery codes, which can be used in adaptive schemes. Actual network simulations and use of these codes for adaptive FEC that can support real-time rate-constrained multimedia applications is part of a future work.

represents the order of the underlying field², N represents the total number of symbols in a single codeword, K represents the number of message symbols in a codeword and Λ_s represents a $2 \times (s+1)$ matrix given by $\begin{bmatrix} N_1 \cdots N_{s+1} \\ K_1 \cdots K_{s+1} \end{bmatrix}$. The entries of matrix Λ_s are constrained by the following equations: $N_i > K_i \forall i \in [1, s]$, $K_i > 0 \forall i \in [1, s]$, $N_{s+1} = K_{s+1}$ and $N = \sum_i N_i$, $K = \sum_i K_i$. Thus Λ_s gives

an s -partition on the set of parity symbols and a $(s+1)$ -partition on the set of message symbols. Figure 2 shows an example of such a code with $s=2$ and $\Lambda_s = \begin{bmatrix} N_1 & N_2 & K_3 \\ K_1 & K_2 & K_3 \end{bmatrix}$. The code is designed such that, $\forall i \in [1, s]$, the pair (N_i, K_i) forms an RS-subcode over $GF(q)$ and the K_{s+1} number of message symbols are transmitted without any protection. Thus the code-graph can be divided into $(s+1)$ disjoint sub-graphs. Obviously such a code graph does not have full density and the density of the overall code has been lowered. It should be noted that an order 1 ($s=1$) PRS code with $N_2 = K_2 = 0$ is equivalent to the traditional full density RS code. In general, a PRS code with $N_{s+1} = K_{s+1} = 0$ does not include any subset of message symbols that are not protected.

We denote the average message throughput due to a single component of the code graph, i.e. a single sub-code by $\rho(N_i, K_i)$. Thus the average total message throughput of an order s PRS code is given by $\tau_m = \sum_{i=1}^{s+1} \rho(N_i, K_i)$. If the number of erasures in a given codeword is equal to L then the average message throughput for a single component is denoted by $\tau_m(L) = \sum_{i=1}^{s+1} \rho(N_i, K_i, L)$. If we assume that all permutations of L erasures are equally likely then the message throughput for a given L due to the subcode i can be expressed by the following equation:

$$\rho(N_i, K_i, L) = \frac{T_1 + T_2}{T_3}, \text{ where } T_3 = K \cdot \binom{N}{L}$$

$$T_1 = \sum_{j=\max(0, N-N_i-L)}^{\min(N_i-K_i, L)} K_j \cdot \binom{N_i}{j} \cdot \binom{N-N_i}{L-j} \text{ and}$$

$$T_2 = \sum_{j=\min(N_i-K_i, L)}^{\min(N_i, L)} \left(\frac{K_i}{N_i} \right) \cdot (N_i - j) \cdot \binom{N_i}{j} \cdot \binom{N-N_i}{L-j}.$$

² In all further discussion we shall drop q from the notation and assume that the order of the field on which the code is based has been pre-specified and is large enough for an MDS code of block-length N to exist. Moreover as multiple codewords form a single FEC block of packets [6], the value of q does not play a significant role.

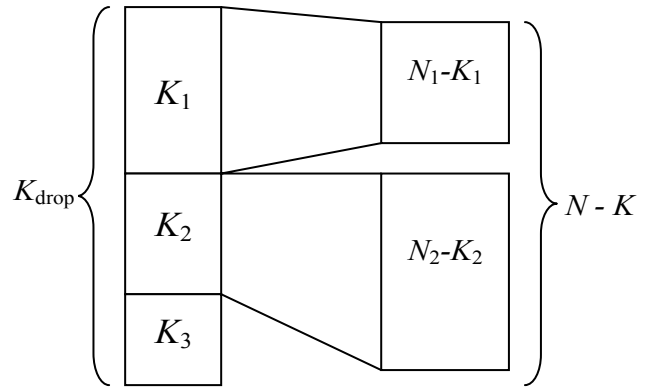


Figure 2 An order 2 PRS code

It should be noted that the above expression could be used to evaluate the average message throughput, $\tau_m(L)$ of the entire PRS code.

IV. PERFORMANCE OF REGULAR PRS CODES

Before proceeding with our performance evaluation results, we emphasize that PRS codes, due to their structure and ability to offer graceful performance degradation [5], may be confused with Prioritized Encoded Transmission (PET) and other unequal error protection schemes. However, it should be clarified that unlike PET, PRS does not make any assumptions about the source. Thus, no priority is assigned to the message symbols and, hence, PRS can render perfectly-regular³ codes capable of partial recovery.

In this section, we evaluate the performance of perfectly-regular PRS codes with $N=160$ and $K=128$. Figure 3 shows the average recovery rendered by regular PRS codes as a function of average number of message symbol loss⁴. We progressively reduce the density of the code by increasing the order by a factor of 2. This is achieved by iteratively partitioning a subcode into two equal partitions. Clearly, when the number of losses exceed the redundancy, higher order PRS codes, despite being completely regular, are capable of partial recovery. It can be observed in Figure 3 that the optimal order to facilitate partial recovery reduces drastically with an increase in the number of losses.

We performed extensive simulations to verify that the drastic decrease in order was a combined consequence of code-regularity and complete protection. In the simulations, we exploited the structure of PRS codes which allow transmission of unprotected message symbols. Careful choice of the number of message symbols⁵ to be transmitted without protection can improve the performance of PRS codes significantly.

Some of the simulation results are outlined in Figure 4 which shows the effect of allowing transmission of unprotected

³ In a regular graph code, all message nodes are connected to an equal number of parity symbols.

⁴ The average number of message symbols erased in a codeword is equal to $((K \cdot L)/N)$ when the total number of erasures are equal to L .

⁵ Note that only the number and not the location of message symbols is to be chosen. This further highlights the difference between PRS and PET.

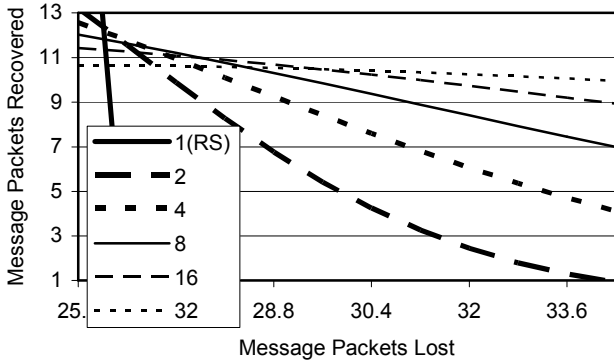


Figure 3. Order dependence of Regular PRS codes.

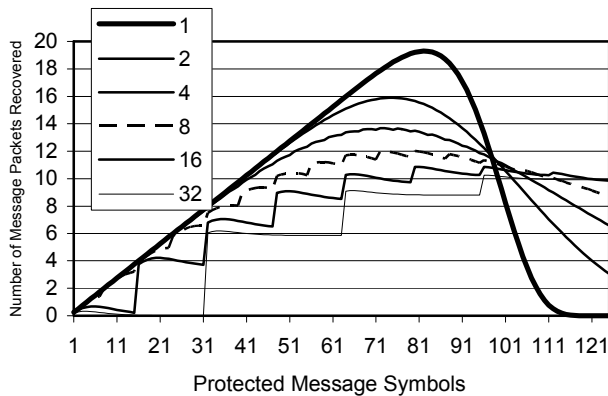


Figure 4. Order dependence of Semi-Regular PRS codes ($L=40$).

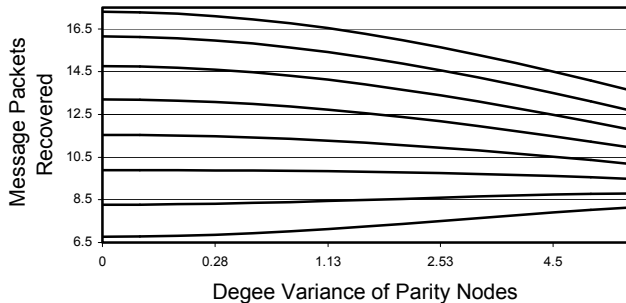


Figure 5. Irregularity Dependence of PRS.

symbols. In this figure, all subcodes are regular with the exception of the unprotected part. Comparing Figure 3 and Figure 4 clearly outlines that partial protection provides significant improvement in the PRS performance. Note that the partial protection, while yielding significant reduction in the order of the optimal code (e.g., reduction from order-32 to order-2), increases the maximum recovery from 10 packets to approximately 20 packets. This is a clear testament to the fact that suitable design of PRS codes can yield significant dividend in the context of overall packet recovery. This packet recovery can be very beneficial for error-resilient multimedia applications. Further results of our simulations are skipped for brevity.

A. Discussion

Figure 4 illustrates that partial protection provides significant improvement in the performance of PRS codes. Partial protection is tantamount to introducing extreme irregularity in a code. Thus, we conclude that irregularity can improve erasure recovery performance. A focal question can be raised here: Can irregularity always improve the erasure recovery performance? We address this question in the following section.

V. IRREGULARITY DEPENDENCE

In this section, we provide experimental results outlining the impact of irregularity on the performance of a constant-density code. Figure 5 shows the effect of irregularity on the average number of message symbols recovered by an order-2 (160,128) PRS code. In this example, the entire message is protected, L varies from 29 to 36 and variance in node-degree is used as an irregularity metric.

Traditionally, irregular constructions of graph codes (e.g., LDPC codes) maintain regularity on one side of the graph, i.e., the message and parity nodes cannot simultaneously have irregular degree distributions. Within the context of complete protection, and while allowing only parity-based irregularity, we obtained results which are outlined in Figure 5. It can be observed that in high loss scenarios, increased irregularity improves erasure recovery. However below a certain loss threshold, regular codes outperform their irregular counterparts. This loss threshold increases as the order of the code is increased. We observed that, with the introduction of partial protection, erasure recovery becomes more sensitive to irregularity. Partial protection is an example of irregularity on both sides of the graph. However, since PRS codes do not allow subcode overlap, irregularity on both sides of the graph implies reduction in the order.

VI. OPTIMAL PRS CODES FOR L ERASURES

Based on the observations in the preceding sections, in this section we design a PRS code to optimize decoding efficiency. The observation deduced so far can be summarized as follows:

- For regular codes, density reduction is necessary to facilitate partial recovery of information.
- Irregularity increases the erasure recovery only beyond a certain loss threshold. This loss threshold decreases with the order.
- Irregularity on one side of the graph-code does not yield significant performance improvements. However, irregularity on both sides renders a reduction in order.
- Partial protection improves the performance of lower order PRS codes.

Hence, we conclude that the optimal PRS codes should be low in order. Such codes should be highly irregular where the irregularity need not be constrained to one side of the graph. Density reduction and increased irregularity can be achieved simultaneously by introducing partial protection. Thus, we conjecture that the optimal (erasure recovery-based) performance is rendered by an order-1 PRS code. Despite extensive simulations with various configurations of (N, K, L) ,

we have not been able to find a counter-example where a higher order PRS code can perform better than the PRS-1 code. Due to brevity constraints, it is not possible to present all the simulation results in this paper. We, nevertheless, provide some sample results based on the exhaustive search.

Let, Ψ_{N,K,K_3} be a set containing all PRS codes of order 1 with $\Lambda_1 = \begin{bmatrix} N_1 & K_2 \\ K_1 & K_2 \end{bmatrix}$ and all PRS codes of order 2 with $\Lambda_2 = \begin{bmatrix} N_1 & N_2 & K_3 \\ K_1 & K_2 & K_3 \end{bmatrix}$. An example of Ψ_{N,K,K_3} is the set $\Psi_{N,K,0}$. In addition to all possible PRS codes of order 1, $\Psi_{N,K,0}$ includes only a subset of all PRS codes of order 2 (i.e., PRS-2). This subset represents PRS-2 codes without any partial protection. In all following figures, the x-axis is the value of $(N_1 - K_1)$, the y-axis is K_1 and the z-axis represents the message throughput of the corresponding code. Note that in

a PRS-1 code, the entire parity is allocated to protect only one subset (with K_1 elements) of the message symbols. Thus, if the code with the maximum message throughput satisfies $N_1 - K_1 = N - K$, then this implies that PRS-1 is the optimal code. It can be observed in the following figures that the optimal code is a PRS -1 code. Thus based on the observations made throughout this paper, we can conclude that PRS-1 codes are the optimal partial recovery codes for a given number of erasures in a codeword.

VII. CONCLUSION

In this paper, we first illustrated the existence of partial recovery codes. We, then, demonstrated that PRS codes provide very efficient (partial) erasure recovery. Furthermore, we showed that the recovery capability of the PRS codes could be improved significantly by altering the degree distribution with respect to the number of erasures. Finally, we conjectured that, for a given number of losses, order-1 PRS codes render optimal (erasure recovery-based) performance.

ACKNOWLEDGMENT

The authors would like to appreciate S. A. Khayam, for his helpful comments in the editing of this document and also for some useful technical discussions related to this work.

The authors would also like to appreciate the help provided by A. Kamath, in generating results for LDPC codes, which helped strengthen the conclusions presented in this work.

REFERENCES

- [1] Y. Wang and Q. Zhu, "Error Control and Concealment for Video Communications: A Review," Proceedings of the IEEE, May 1998.
- [2] Y. Wang, M. Hannuksela, V. Varsa, A. Hourunranta, and M. Gabbouj, "The Error Concealment Feature in H.26L Test Model," IEEE ICIP, September 2002.
- [3] S. A. Khayam, S. S. Karande, H. Radha, and D. Loguinov, "Performance Analysis and Modeling of Errors and Losses over 802.11b LANs for High-Bitrate Real-Time Multimedia," Signal Processing: Image Communication - Special Issue on Recent Advances in Wireless Video, vol.18, no.7, pp. 575-595, August 2003
- [4] M. Yajnik, J. Kurose, D. Towsley, "Packet Loss Correlation in the Mbone Multicast Network," Proceedings of IEEE Global Internet Conference, November 1996.
- [5] S. S. Karande and H. Radha, "Partial Reed Solomon Codes," IEEE Information Theory Workshop, 2003.
- [6] S. S. Karande and Hayder Radha, "A New Family of Channel Coding Schemes for Real-Time Visual Communications," ICME, July 2003.
- [7] L. Rizzo, "Effective erasure codes for reliable computer communication protocols," ACM Computer Communication Review, April 1997.
- [8] M. G. Luby, M. Mitzenmacher, M. A. Shokrollahi, and D. A. Spielman, "Efficient Erasure Correcting Codes," IEEE Transactions on Information Theory, vol. 47, pp 569 - 584, February 2001.
- [9] M. G. Luby, M. Mitzenmacher, M. A. Shokrollahi, and D. A. Spielman, "Improved Low-Density Parity-Check Codes Using Irregular Graphs," IEEE Transactions on Information Theory, vol. 47, pp 585 - 598, February 2001.
- [10] J. C. Bolot, S. Fosse-Parisis, and D. Towsley, "Adaptive FEC-based error control for Internet telephony," Proceedings of IEEE INFOCOM
- [11] R. E. Blahut, "Theory and Practice of Error Control Codes," Addison-Wesley, May 1984.

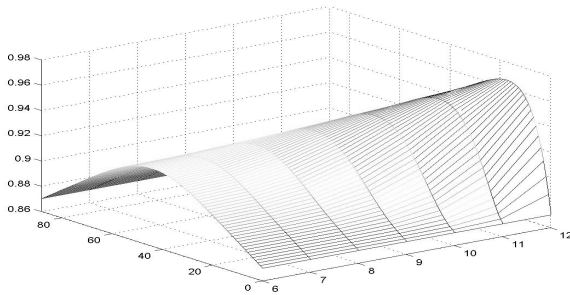


Figure 6. Message Throughput ($N = 100, K = 88, L = 13$).

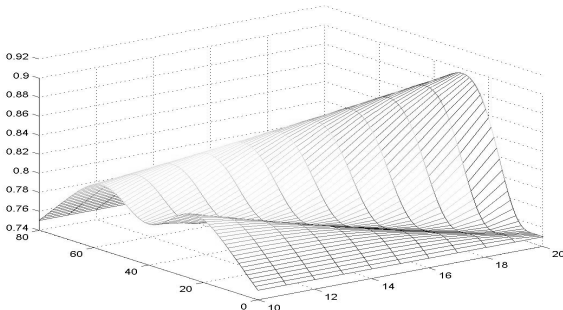


Figure 7. Message Throughput ($N = 100, K = 80, L = 25$).

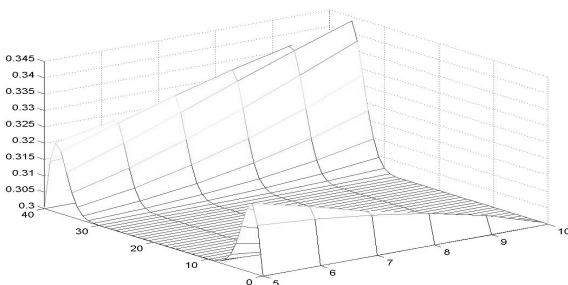


Figure 8. Message Throughput ($N = 50, K = 40, L = 35$).