Module 2:
Fundamental Behavior of Electrical Systems

2.0 Introduction

All electrical systems, at the most fundamental level, obey Maxwell's equations and the postulates of electromagnetics. Under certain circumstances, approximations can be made that allow simpler methods of analysis, such as circuit theory, to be employed. However, the problems associated with EMC usually involve departures from these approximations. Therefore, a review of fundamental concepts is the logical starting point for a proper understanding of electromagnetic compatibility.

2.1 Fundamental quantities and electrical dimensions

- **speed of light, permittivity and permeability in free space**

The speed of light in free space has been measured through very precise experiments, and extremely accurate values are known (299,792,458 m/s). For most purposes, however, the approximation

$$c \approx 3\text{x}10^8 \text{ m/s}$$

is sufficiently accurate. In the International System of units (SI units), the constant known as the *permeability of free space* is defined to be

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m.}$$

The constant *permittivity of free space* is then derived through the relationship

$$c = \frac{1}{\sqrt{\varepsilon_o \mu_o}}$$

and is usually expressed in SI units as

$$\varepsilon_0 = \frac{1}{c^2 \mu_o} \cong 8.854 \times 10^{-12} \cong \frac{1}{36\pi} \times 10^{-9} \text{ F/m.}$$

Both the permittivity and permeability of free space have been repeatedly verified through experiment.

- **wavelength in lossless media**

Wavelength λ is defined as the distance between adjacent equiphase points on a wave. For an electromagnetic wave propagating in a lossless medium, this is given by

$$\lambda = \frac{v}{f}$$

where f is frequency. In free space $v=c$, and in a simple lossless medium other than free space, the velocity of propagation is $v = 1\big/\sqrt{\mu\varepsilon}$, where $\varepsilon = \varepsilon_r\varepsilon_0$ and $\mu = \mu_r\mu_0$. Here are ε_r and μ_r are known as the *relative permittivity* and *relative permeability* of the medium, respectively. These will be discussed in greater detail later in this chapter. For now it is sufficient to note that as the values of permittivity and permeability increase, velocity of propagation decreases, and therefore wavelength decreases (thus the wavelength at a particular frequency is shorter in a material than in free space). The free space wavelengths associated with waves of various frequencies are shown below:

frequency	wavelength
60 Hz	5000 km (3107 mi)
3 kHz	100,000 m
30 kHz	10,000 m
300 kHz	1000 m
3 MHz	100 m
30 MHz	10 m
300 MHz	1 m
3 GHz	10 cm
30 GHz	1 cm
300 GHz	0.1 cm

- **relationship between physical and electrical dimensions**

The size of an electrical circuit or circuit component, as compared to a wavelength, determines, to a certain extent, the manner in which it interacts with EM fields. For instance, in order for an antenna to effectively receive and transmit signals at a certain frequency, it must be a significant fraction of a wavelength long at that frequency. Likewise, other types of electrical components may emit or receive interference-causing signals if they are large compared to a wavelength. In addition, the electrical characteristics of a circuit component are often very different when it is electrically large (i.e. the frequency of operation is high) than when it is electrically small (the frequency of operation is low). Kirchoff's voltage and current laws are only valid if the circuit elements under consideration are small compared to a wavelength. If the components under consideration are electrically large, then Maxwell's equations must be applied in order to analyze device behavior.

The electrical dimensions of a device or circuit are determined by comparing physical dimensions to wavelength. A device with length l has electrical dimensions (in wavelengths)

$$d_e = \frac{l}{\lambda}.$$

The electrical dimensions of a circuit are determined by first calculating the wavelength at the highest frequency of interest, and then determining d_e. Devices or circuits are considered to be electrically small if the largest dimension is much smaller than a wavelength ($kd_e \ll 1$). Typically structures which are less than one tenth of a wavelength long are considered to be electrically small. It must be remembered that the electrical dimensions are dependent upon the material in which EM waves propagate. A device may be electrically much larger when it is embedded in a printed circuit board than it is when surrounded by air. Likewise, a capacitor which contains a high permittivity dielectric is electrically larger than a similar capacitor whose plates are separated by air.

2.2 Common EMC units of measure

Commonly measured EMC quantities may take on values that range over many orders of magnitude. As a result, it is often convenient to express these quantities in *decibels*. The quantity most commonly expressed in decibels is power gain. The power gain of an amplifier is the ratio of output power to input power

$$\text{power gain} = \frac{P_{out}}{P_{in}}$$

which may be expressed in decibels as

$$\text{power gain}_{\text{dB}} \equiv 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right).$$

Voltage gain and current gain are also sometimes represented in terms of decibels. If the input and output powers of an amplifier are dissipated by two equal resistances then the power gain in decibels is

$$\text{power gain}_{\text{dB}} \equiv 10\log_{10}\left(\frac{|V_{out}|^2 / R}{|V_{in}|^2 / R}\right) = 10\log_{10}\left(\frac{|I_{out}|^2 R}{|I_{in}|^2 R}\right).$$

From this it is seen that voltage gain in decibels is represented by

$$\text{voltage gain}_{\text{dB}} \equiv 20\log_{10}\left(\frac{|V_{out}|}{|V_{in}|}\right)$$

while current gain is expressed as

$$\text{current gain}_{\text{dB}} = 20\log_{10}\left(\frac{|I_{out}|}{|I_{in}|}\right).$$

It is important to note that decibels represent the ratio of two quantities, or more precisely, the value of one quantity as referenced to some base quantity. The units *dB* describe power in watts referenced to 1 watt

$$\text{power in dB} \equiv 10\log_{10}\left(\frac{watts}{1W}\right)$$

while the units *dBm* describe power in watts referenced to 1 milliwatt

$$\text{power in dBm} \equiv 10\log_{10}\left(\frac{watts}{1mW}\right).$$

In each of these cases, a positive value of dB or dBm indicates that the power in watts is greater than the reference quantity, while a negative value of dB or dBm indicates that the power in watts is less than the reference quantity. Other quantities are sometimes expressed in decibels, including

$$\text{dB}\,\mu\text{W} \equiv 10\log_{10}\left(\frac{watts}{1\mu W}\right)$$

$$\text{dBmA} \equiv 20\log_{10}\left(\frac{amps}{1mA}\right)$$

$$\text{dBmV} \equiv 20\log_{10}\left(\frac{volts}{1mV}\right).$$

2.3 Linear systems

- **linear system response**

Many complex systems obey the *principle of superposition*, which permits a complicated problem to be decomposed into a number of simpler problems. Superposition is often used to analyze electric circuits which have more than one independent voltage or current source. The voltage or current at any point in certain circuits may be determined by summing the voltages or currents due to each independent source individually. Superposition is also frequently used to determine electric and magnetic fields in regions where multiple sources exist.

Application of the principle of superposition requires that the system under consideration be *linear*. Consider a system with input $x(t)$, and output $y(t)$. If

$$y_1(t) = L\{x_1(t)\}$$

and

$$y_2(t) = L\{x_2(t)\}$$

then the system operator $L\{\sim\}$ is said to be linear if

$$L\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 y_1(t) + a_2 y_2(t)$$

where a_1 and a_2 are constant scale factors. Linearity requires that the differential or integral operator $L\{\sim\}$ does not involve products of the dependent variables or their derivatives among themselves. Thus, the systems

$$y(t) = \sin[x(t)]$$

and

$$y(t) = x^2(t)$$

are clearly nonlinear.

2.4 Review of electromagnetics

- **volume source densities**

Electromagnetic fields are supported by electric charges, which may be at rest or in motion. These charges may be free, such as those exiting in free space or in a conductive material, or bound, such as those which may exist in a dielectric or magnetic material. The fields discussed in this course will be those which are supported by large aggregates of charge. The macroscopic field representations developed are best described in terms of space-time average source density functions, which exist over regions and periods that are large compared with the atomic regime, but small on a laboratory scale. Effects which occur on a microscopic or quantum-mechanical level are outside the scope of this course.

Four source quantities which may support time varying electric and magnetic fields are used throughout this course. Free charges are represented in terms of the *volume density of charge*

$$\rho(\vec{R}, t) = \lim_{\Delta v \to 0^+} \frac{\sum_{q_i \in \Delta v} q_i}{\Delta v} \qquad (C/m^3)$$

where $\vec{R} = \hat{x}x + \hat{y}y + \hat{z}z$ is a general three-dimensional position vector, and q_i are discrete charges which reside within a volume of space Δv. Free currents (composed of moving free charges) are described in terms of a *volume density of current*

$$\vec{J}(\vec{R},t) = \lim_{\Delta v \to 0^+} \frac{\sum_{q_i \in \Delta v} q_i \vec{u}_i}{\Delta v} \qquad (A/m^2)$$

where $\vec{u}_i$ is the velocity of the ith discrete charge. Bound polarization charges may exist within, or on the surface of dielectric materials. Such charges are described by a *volume density of polarization*

$$\vec{P}(\vec{R},t) = \lim_{\Delta v \to 0^+} \frac{\sum_{\vec{p}_i \in \Delta v} \vec{p}_i}{\Delta v} \qquad (C/m^2)$$

where $\vec{p}_i = q_i \vec{d}$ is the moment associated with the ith dipole which exists within a volume of space Δv. Finally, bound magnetic currents, such as those that may exist within certain types of magnetic materials, are represented by a *volume density of magnetization*

$$\vec{M}(\vec{R},t) = \lim_{\Delta v \to 0^+} \frac{\sum_{\vec{m}_i \in \Delta v} \vec{m}_i}{\Delta v} \qquad (A/m).$$

In this expression $\vec{m}_i = I\hat{n}A$ is the magnetic moment associated with a current I circulating around a small loop bounding cross-sectional area A.

- **continuity equation**

The principle of conservation of charge states that charge may be neither created nor destroyed. The total current flowing out of a volume of space is the total outward flux of the current density through the surface bounding that volume. If conservation of charge is obeyed, then the current flowing out of the volume is equal to the rate of decrease of charge contained within the volume, thus

$$I = \oint_S \vec{J} \cdot \vec{ds} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho \, dv \ .$$

Application of the divergence theorem yields

$$\int_V \nabla\cdot\vec{J}\,dv = -\int_V \frac{\partial\rho}{\partial t}dv .$$

This relationship must be satisfied for any volume V**,** therefore

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t} .$$

This expression is known as the *continuity equation*, and states simply that the density of current flowing away from a point must be equal to the time rate of decrease of charge at that point.

- **Lorentz force equation**

In the presence of time varying electric and magnetic fields, the force experienced by a charged particle q moving with velocity $\vec{u}$ is

$$\vec{F} = q(\vec{E} + \vec{u}\times\vec{B}) .$$

This expression is known as the *Lorentz force equation*. The field quantities $\vec{E}(\vec{R},t)$ (*electric field intensity*) and $\vec{B}(\vec{R},t)$ (*magnetic flux density*) are defined through the Lorentz force equation. The electric field intensity is defined by

$$\vec{E} = \left.\frac{\vec{F}_{q_1}}{q_1}\right|_{\vec{u}_1 = 0}$$

and magnetic flux density is defined by

$$\vec{u}_2\times\vec{B} = \left.\left(\frac{\vec{F}_{q_2}}{q_2} - \vec{E}\right)\right|_{\vec{u}_2 \neq 0} .$$

- **Maxwell's equations in free space**

- large-scale (integral) form

All electromagnetic phenomena are described by *Maxwell's equations*. These four equations may be represented in either large-scale (integral) form, or in point (differential) form. When describing interactions involving finite objects with specific shapes and boundaries it is convenient to express Maxwell's equations in large-scale form.

i. Faraday's law

The large-scale form of *Faraday's law*

$$\oint_C \vec{E}\cdot dl = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$$

states that a voltage is induced in a conducting loop that lies along a contour *C*, which is either stationary and immersed in a time changing magnetic field, or moving in a static magnetic field. This voltage is proportional to the time rate of change of magnetic flux which penetrates the surface bounded by the loop. The line integral term on the left hand side of Faraday's law is frequently known as the *electromotive force* or *emf*

$$emf = \oint_C \vec{E}\cdot d\vec{l}\,.$$

The total magnetic flux through the surface S which is bounded by contour C is

$$\psi_m = \int_S \vec{B}\cdot d\vec{s}$$

therefore Faraday's law may be written

$$emf = -\frac{d\psi_m}{dt}\,.$$

If no time-dependent magnetic flux penetrates the surface S, then the induced emf is zero (Kirchoff's voltage law). It should be remembered that the induced emf is a distributed quantity, not a localized source (although for electrically small circuits it may be modeled as such). If an open-circuited loop is placed in a time-changing magnetic field, an emf is still induced. Although no current flow occurs in this case, the emf manifests itself as a potential difference which appears across the terminals of the loop.

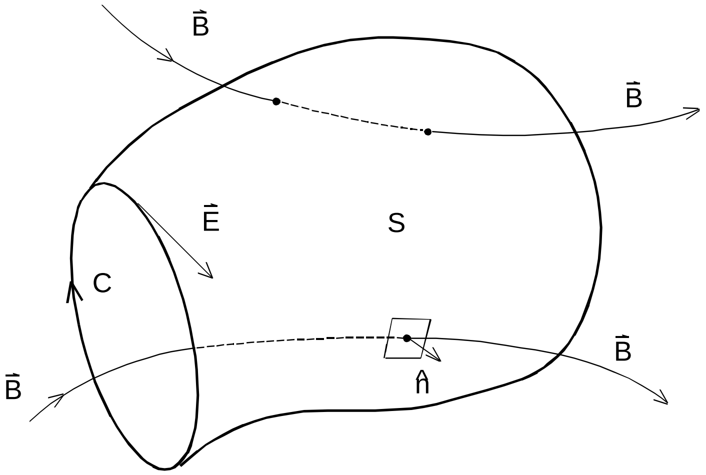

Figure 1.Illustration of Faraday's law.

ii. Ampere's law

Ampere's law states that the circulation of magnetic flux density around any closed contour in free space is equal to the total current flowing through a surface bounded by that contour times the permeability of free space

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_o \int_S \vec{J}\cdot d\vec{s} + \mu_o \varepsilon_o \frac{d}{dt}\int_S \vec{E}\cdot d\vec{s} .$$

This indicates that a time-changing electric field gives rise to a magnetic field. The line integral term on the left-hand side of this expression is known as the *magnetomotive force* or *mmf*

$$mmf = \frac{1}{\mu_o}\oint_C \vec{B}\cdot d\vec{l} .$$

The first term on the right side of the Ampere's law expression represents the total current flowing through the surface S bounded by the contour C due to free charges, and is referred to as *conduction current*

$$I_c = \int_S \vec{J} \cdot \vec{ds} .$$

The second term on the right side of the Ampere's law expression is the total *displacement current* flowing through the surface S bounded by the contour C

$$I_d = \varepsilon_o \frac{d}{dt} \int_S \vec{E} \cdot \vec{ds} .$$

Therefore, Ampere's law may be expressed

$$mmf = I_c + I_d .$$

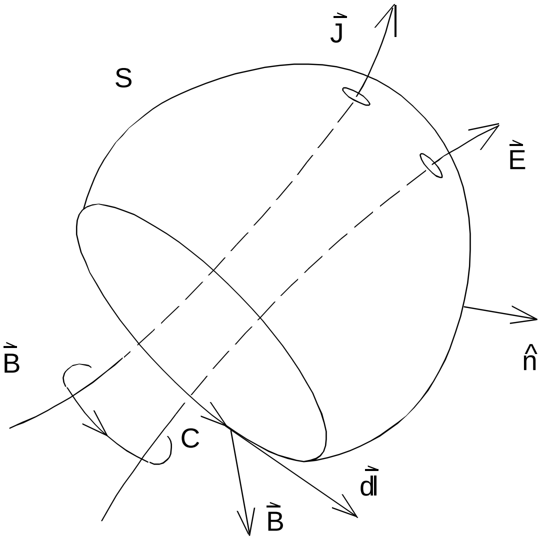

Figure 2.Illustration of Ampere's law.

iii. Gauss's law

The large-scale form of *Gauss's law* states that total electric field flux through a closed surface S is equal to the amount of charge enclosed by that surface divided by the permittivity of the surrounding medium. In free space, this is

$$\oint_S \vec{E} \cdot \vec{ds} = \frac{1}{\varepsilon_o} \int_V \rho \, dv .$$

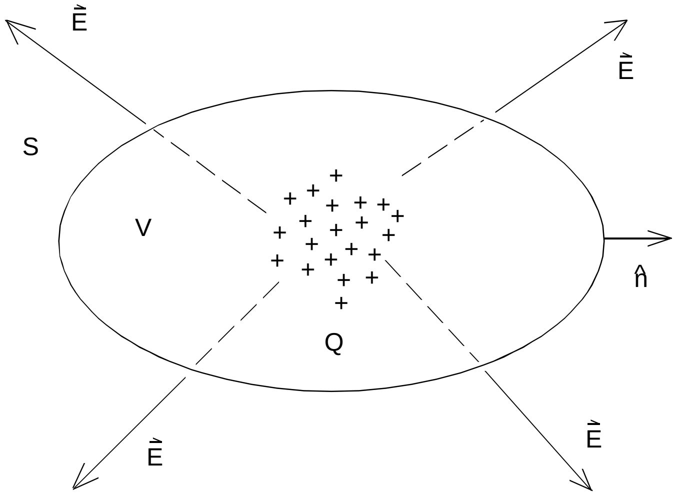

Figure 3.Illustration of Gauss's law.

iv. Magnetic source law

The last of Maxwell's equations

$$\oint_S \vec{B}\cdot\vec{ds} \ = \ 0$$

is the *magnetic source law,* which states that the integral of magnetic flux density over any closed surface vanishes. This indicates that the total magnetic flux flowing out of a closed surface is zero (magnetic field lines from closed loops). This reflects the fact that magnetic source charges have yet to be observed in nature.

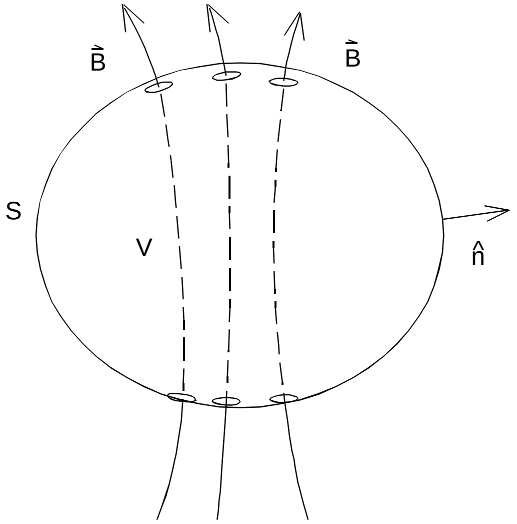

Figure 4.Illustration of magnetic source law.

- point form

When describing interactions at a single point in space, it is convenient to express Maxwell's equations in point, or differential form. Point form representations of Faraday's law and Ampere's law are obtained by applying Stoke's theorem to the large-scale forms of these expressions. This results in the relationships

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{B} = \mu_o\left(\vec{J} + \varepsilon_o\frac{\partial\vec{E}}{\partial t}\right).$$

The point form expressions for Gauss's law and the magnetic source law are obtained through application of the divergence theorem to the large-scale expressions of these relationships

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_o}$$

$$\nabla\cdot\vec{B} = 0.$$

- Maxwell's equations in materials

In the presence a material, Maxwell's equations must be modified to include sources which arise from polarization and magnetization effects. Including these leads to a system of Maxwell's equations in terms of all EM sources

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{B} = \mu_o\left(\vec{J} + \frac{\partial\vec{P}}{\partial t} + \nabla\times\vec{M} + \varepsilon_o\frac{\partial\vec{E}}{\partial t}\right)$$

$$\nabla\cdot\vec{E} = \frac{(\rho - \nabla\cdot\vec{P})}{\varepsilon_o}$$

$$\nabla\cdot\vec{B} = 0\ .$$

Here the source term

$$\vec{J}_{total} = \vec{J} + \frac{\partial\vec{P}}{\partial t} + \nabla\times\vec{M} + \varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

represents the total effective current density that may exist in a medium with dielectric and magnetic properties, and

$$\rho_{total} = \rho - \nabla\cdot\vec{P}$$

is the total effective charge density in such a medium.

- **constitutive parameters of a medium**

  - **electric flux density and permittivity**

In the presence of a dielectric medium the total source charge density consists of free charge and bound equivalent polarization charge. In this case the point form of Gauss's law becomes

$$\nabla\cdot(\varepsilon_o\vec{E} + \vec{P}) = \rho\ .$$

It is convenient to define an auxiliary field quantity, the *electric flux density* which accounts for effects due to the presence of the medium

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}.$$

In a linear, isotropic medium, polarization is directly proportional to electric field intensity

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

where χ_e is known as electric susceptibility. Substitution yields

$$\begin{aligned} \vec{D} &= \varepsilon_o (1 + \chi_e) \vec{E} \\ &= \varepsilon_o \varepsilon_r \vec{E} = \varepsilon \vec{E} \end{aligned}$$

where

$$\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_o}$$

is known as the *relative permittivity*, or *dielectric constant* of a medium and ε is known as the permittivity.

- **magnetic field intensity and permeability**

In the presence of a magnetic medium the total source current density consists of free current and equivalent magnetization current. In this case the point form of Ampere's law becomes

$$\nabla \times \left(\frac{\vec{B}}{\mu_o} - \vec{M} \right) = \vec{J} + \frac{\partial}{\partial t} (\varepsilon_o \vec{E} + \vec{P})$$

It is convenient to define a second auxiliary field, the *magnetic field intensity*, which accounts for effects due to the presence of a magnetic material

$$\vec{H} = \frac{\vec{B}}{\mu_o} - \vec{M}.$$

In a linear, isotropic medium, magnetization is directly proportional to magnetic field intensity

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is known as magnetic susceptibility. Substitution yields

$$\vec{B} = \mu_o(1+\chi_m)\vec{H}$$

$$= \mu_o\mu_r\vec{H} = \mu\vec{H}$$

where

$$\mu_r = 1+\chi_m = \frac{\mu}{\mu_o}$$

is known as the *relative permeability* of a medium and μ is the permeability.

- **independent Maxwell's equations**

It is important to note of Maxwell's equations that only Faraday's law and Ampere's law are independent. The equations which involve the divergence operation can be derived from the curl operations. Consider the point form of Ampere's law

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}.$$

Taking the divergence of both sides leads to

$$\nabla\cdot(\nabla\times\vec{H}) = \nabla\cdot\vec{J} + \frac{\partial}{\partial t}\nabla\cdot\vec{D}.$$

By vector identity, the divergence of the curl of a vector is zero, therefore

$$\nabla\cdot\vec{J} = -\frac{\partial}{\partial t}\nabla\cdot\vec{D}.$$

Application of the continuity equation results in

$$-\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial t}\nabla\cdot\vec{D} = 0.$$

Combining terms under the differential operator yields

$$\frac{\partial}{\partial t}(\nabla\cdot\vec{D} - \rho) = 0 .$$

The term in parenthesis must be equal to a constant

$$\nabla\cdot\vec{D} - \rho = C(\vec{r}) .$$

It is argued that the temporal constant in the above expression is zero, therefore

$$\nabla\cdot\vec{D} = \rho .$$

In a similar way the magnetic source law may be derived from Ampere's law.

- **Ohm's law and conduction current density**

Conduction currents are caused by the drift motion of charged particles under the influence of an electric field. The total current density may be composed of various charge species drifting with different velocities, therefore

$$\vec{J} = \sum_i N_i q_i \vec{u}_i$$

where N_i is the particle number density, q_i is the particle charge, and $\vec{u}_i$ is the average ensemble drift velocity. In a metallic conductor where only electrons are present

$$\vec{u} = -\mu_e \vec{E}$$

where μ_e is the electron mobility. This leads to

$$\vec{J} = -\rho_e \mu_e \vec{E} = \sigma\vec{E}$$

which is known as *Ohm's law*. The conductivity, σ, of a material is defined as

$$\sigma = -\rho_e \mu_e$$

where ρ_e is electronic charge density, and μ_e is electron mobility.

- **displacement current density**

Prior to the formulation of Maxwell's equations, the point form of Ampere's law was stated

$$\nabla\times\vec{H} = \vec{J}\,.$$

This, however, is not consistent with the principle of conservation of charge. This can be seen by taking the divergence of both sides of Ampere's law

$$\nabla\cdot(\nabla\times\vec{H}) = 0 = \nabla\cdot\vec{J}$$

The divergence of $\vec{J}$ is in general not equal to zero. The above equation must be modified to include time variation of charge

$$\nabla\cdot(\nabla\times\vec{H}) = 0 = \nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t}$$

or

$$\nabla\cdot(\nabla\times\vec{H}) = 0 = \nabla\cdot\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)$$

Thus the form of Ampere's law which is consistent with conservation of charge is

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

Here the additional term $\partial\vec{D}/\partial t$ is known as the displacement current density. In low frequency systems, the displacement current is often negligible. However as frequency increases, effects due to displacement current become more pronounced, such as in the dielectric of a capacitor which has a rapidly changing voltage applied to it.

- **relationships between field quantities and potential functions**

The magnetic source law states that a steady magnetic field is solenoidal with

$$\nabla\cdot\vec{B} = 0\,.$$

Using the vector identity that the divergence of the curl of any vector field is always zero

$$\nabla\cdot(\nabla\times\vec{A}) \equiv 0$$

it is apparent that the steady magnetic field may be represented as the curl of a vector

$$\vec{B} = \nabla\times\vec{A}$$

where $\vec{A}$ is known as the *EM vector potential*. The point form of Faraday's law states

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\nabla\times\frac{\partial\vec{A}}{\partial t}.$$

This may be written as

$$\nabla\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0.$$

Using the vector identity that the curl of the gradient of any scalar field is always zero

$$\nabla\times(\nabla\varphi) \equiv 0$$

it is seen that the quantity in parenthesis may be represented as the gradient of a scalar

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\nabla\varphi$$

where the negative gradient is chosen so that φ, the *EM scalar potential*, is consistent with the static potential difference. Finally, electric filed is expressed in terms of the EM potential functions as

$$\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}.$$

- **wave equations**

In a source-free region of a simple medium, the point forms of Maxwell's equations specialize to

$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\nabla\times\vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\cdot\vec{E} = 0$$

$$\nabla\cdot\vec{H} = 0\ .$$

Taking the curl of both sides of Faraday's law yields

$$\nabla\times\nabla\times\vec{E} = -\mu\frac{\partial}{\partial t}(\nabla\times\vec{H}) = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

Applying the vector identity

$$\nabla\times\nabla\times\vec{E} = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\nabla^2\vec{E}$$

leads to the homogeneous electric field wave equation

$$\nabla^2\vec{E} - \frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

where $v = 1/\sqrt{\mu\varepsilon}$ is the velocity of light in the simple medium. A similar development on Ampere's law yields the homogeneous magnetic field wave equation

$$\nabla^2\vec{H} - \frac{1}{v^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0\ .$$

- **boundary conditions**

At the surface of a perfect conductor the tangential component of electric field vanishes

$$\hat{t}\cdot\vec{E} = E_t = 0 \, .$$

This condition yields unique solutions to Maxwell's equations. The surface charge and surface current are subsequently related to the surface fields as

$$\rho_s = \hat{n}\cdot\vec{D} = \varepsilon_o(\hat{n}\cdot\vec{E}) = \varepsilon_o E_n$$

and

$$\vec{J}_s = \hat{n}\times\vec{H} \, .$$

When layers of differing dielectric materials exist, the tangential components of electric and magnetic field must be continuous across the interface

$$\hat{n}\times(\vec{E}_1 - \vec{E}_2) = 0$$

$$E_{t1} = E_{t2}$$

$$\hat{n}\times(\vec{H}_1 - \vec{H}_2) = \vec{J}_s$$

$$H_{t1} - H_{t2} = \vec{J}_s$$

which again leads to unique solutions of Maxwell's equations. The normal field components are then related by

$$\hat{n}_2\cdot(\vec{D}_1 - \vec{D}_2) = \rho_s$$

and

$$B_{n1} = B_{n2} \, .$$

- **Poynting's power theorem**

Energy is transported by electromagnetic waves. The power balance relation for EM fields is known as *Poynting's theorem.* This theorem describes the relationship between power (rate of energy transfer) and electric and magnetic field intensities. Consider the Maxwell curl equations

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}.$$

According to a vector identity

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

therefore

$$\nabla\cdot(\vec{E}\times\vec{H}) = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} - \vec{E}\cdot\vec{J}.$$

In a linear, isotropic, homogeneous medium

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \vec{H}\cdot\frac{\partial(\mu\vec{H})}{\partial t} = \frac{1}{2}\frac{\partial(\mu\vec{H}\cdot\vec{H})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu|\vec{H}|^2\right)$$

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \vec{E}\cdot\frac{\partial(\varepsilon\vec{E})}{\partial t} = \frac{1}{2}\frac{\partial(\varepsilon\vec{E}\cdot\vec{E})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon|\vec{E}|^2\right)$$

and

$$\vec{E}\cdot\vec{J} = \vec{E}\cdot(\vec{J}^{\,s}+\vec{J}^{\,i}) = \vec{E}\cdot(\sigma\vec{E}+\vec{J}^{\,i}) = \sigma|\vec{E}|^2+\vec{E}\cdot\vec{J}^{\,i}$$

where $\vec{J}^{\,i}$ is the impressed current and $\vec{J}^{\,s} = \sigma\vec{E}$ is the secondary current. Thus

$$\nabla\cdot(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon|\vec{E}|^2 + \frac{1}{2}\mu|\vec{H}|^2\right) - \sigma|\vec{E}|^2 - \vec{J}^{\,i}\cdot\vec{E}.$$

Integrating over a volume of space and applying the divergence theorem yields

$$-\int_V \vec{J}^{\,i}\cdot\vec{E}\,dV + \oint_S (\vec{E}\times\vec{H})\cdot d\vec{s} \;=\; -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon|\vec{E}|^2 + \frac{1}{2}\mu|\vec{H}|^2\right) dv - \int_V \sigma|\vec{E}|^2\,dv\,.$$

where the power delivered by the sources to the fields is $-\int_V \vec{J}^{\,i}\cdot\vec{E}\,dV$. It is recognized that the first term on the right side of this expression represents the time rate of change of EM energy (sum of stored electric and stored magnetic energy). The second term on the right side represents the ohmic power dissipated due to conduction currents flowing in the presence of the electric field. The terms on the right side must equal the power leaving a volume of space through the surface bounding that volume. The quantity $\vec{E}\times\vec{H}$ must therefore represent power flow per unit area. A power density vector known as the *Poynting vector* may be defined

$$\vec{P} = \vec{E}\times\vec{H} \qquad watts/m^2$$

which is associated with an electromagnetic field. The Poynting vector lies in a direction normal to both $\vec{E}$ and $\vec{H}$. If the region of interest is lossless, then $\sigma = 0$ and the total power flowing into a volume bounded by a closed surface is equal to the rate of increase of stored electric and magnetic energies. When the fields are static, the total power flowing into the volume is equal to the ohmic power dissipated in the volume.

- **definitions of resistance, capacitance and inductance**

Consider two conductors of arbitrary shape, which are immersed in a dielectric medium. A potential difference V maintains a charge separation of +Q and -Q on the conductors.

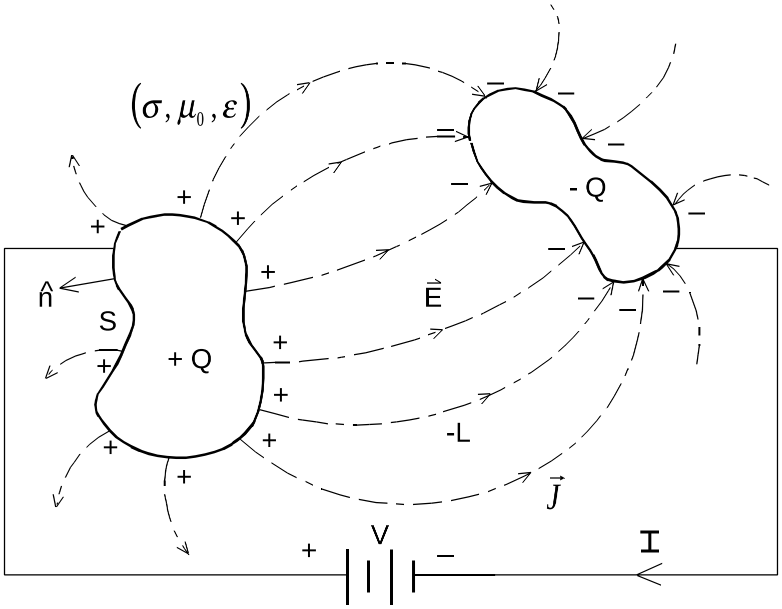

Figure 5.Arbitrary conductors immersed in a dielectric medium.

Using this model, basic relationships for resistance and capacitance in terms of field quantities may be determined.

- **resistance**

If the dielectric medium separating the conductors has a small non-zero conductivity (referred to as a *lossy* medium), then a current will flow in the material. The resistance encountered by this current is expressed by

$$R = \frac{V}{I} = \frac{-\int_L \vec{E}\cdot d\vec{l}}{\oint_S \vec{J}\cdot d\vec{s}} .$$

- **capacitance**

The expression for the Coulomb potential maintained by a surface charge distribution states

$$V = \frac{1}{4\pi\varepsilon_o}\int_S \rho_s \frac{1}{|\vec{R}-\vec{R}^{\,\prime}|}\, ds^{\prime} .$$

For a point on the surface, it is seen that the potential of an isolated conductor is directly proportional to the total charge which maintains that potential. For an isolated system of conductors, the ratio of charge to potential remains constant. The constant of proportionality relating potential to charge is known as capacitance. This relationship is written

$$Q = CV$$

where Q is charge, V is potential, and C is capacitance. A capacitor consists of two conductors separated by an EM medium. A voltage source causes a charge separation to occur, resulting in the formation of an electric field between the conductors. It should be remembered that the capacitance of a capacitor is a physical property of a system of conductors, depending upon the geometry of the system and the permittivity of the medium between the conductors. Capacitance does *not* depend on the amount of charge deposited on the conductors, or the potential difference which maintains the charge separation. Thus a capacitor has a capacitance even when no potential difference exists between the conductors, and no charge is present on the conductors. In general capacitance is represented by

$$C = \frac{Q}{V} = \frac{\int_S \rho_s ds}{-\int_L \vec{E} \cdot \vec{dl}} = \frac{\int_S \vec{D} \cdot \vec{ds}}{-\int_L \vec{E} \cdot \vec{dl}} .$$

Using $\vec{J} = \sigma\vec{E}$ and $\vec{D} = \varepsilon\vec{E}$ we see that $RC = \frac{\varepsilon}{\sigma}$.

- **inductance**

Consider two closed contours C_1, and C_2 which bound surfaces S_1, and S_2, respectively. A current I_1 flowing along C_1 will give rise to a magnetic field. A certain fraction of the magnetic flux associated with this field will penetrate (link with) surface S_2.

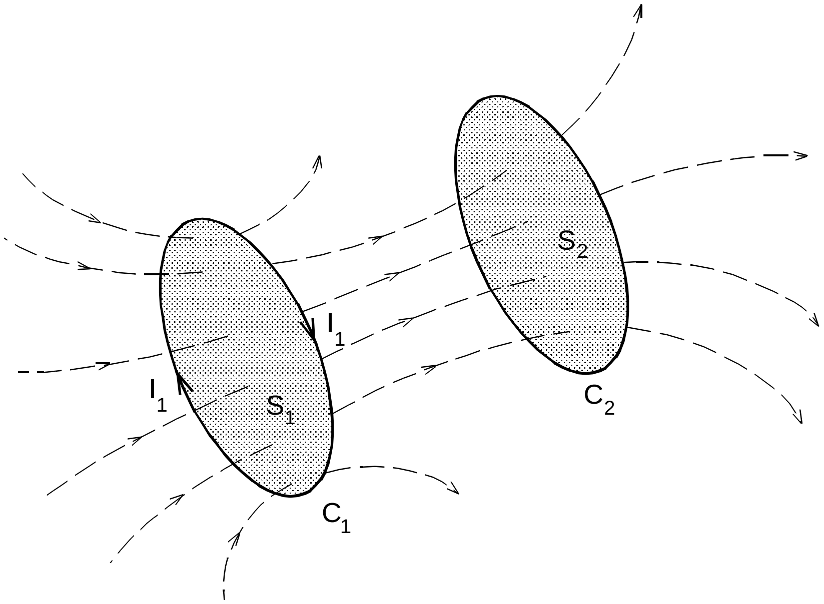

Figure 6.Magnetically coupled circuits.

The mutual flux is therefore that portion of the flux form the field associated with loop 1 that passes through loop 2

$$\Phi_{21} = \int_{S_2} \vec{B}_1 \cdot \vec{ds}_2$$

The magnitude of the magnetic flux density is proportional to current, therefore, the mutual flux Φ_{21} is also proportional to current I_1

$$\Phi_{21} = L_{21} I_1$$

where the constant of proportionality L_{12} is called the mutual inductance between loops C_1 and C_2. If the loop C_2 has N_2 turns, then a flux linkage Λ_{21} is defined

$$\Lambda_{21} = N_2 \Phi_{21}$$

which also may be represented by

$$\Lambda_{21} = L_{21} I_1$$

or

$$L_{21} = \frac{\Lambda_{21}}{I_1} .$$

From this it is evident that the mutual inductance between two circuits is the amount of magnetic flux linkage with one circuit per unit current in the other. Self inductance is the magnetic flux linkage per unit current within the circuit itself

$$L_{11} = \frac{\Lambda_{11}}{I_1} .$$

It should be noted that the self inductance of a circuit depends on the shape of the circuit and configuration of conductors within the circuit, as well as the permeability of the medium in which the circuit is immersed. If the surrounding medium is linear, then self inductance does not depend on the amount of current flowing within the circuit. Self inductance exists regardless of whether the circuit is open or closed or whether it is located near another circuit.

- **time harmonic field representations**

The way that the field quantities $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$ vary with time depends on the nature of source functions $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$. In general EM sources have an arbitrary dependence on time. In certain cases, however, it is easier assume that the source quantities have a sinusoidal time dependence. Because Maxwell's equations are linear, sinusoidal variations of the source terms at a certain frequency will result in sinusoidal variations of the field quantities at the same frequency (when the system is in steady state). Assuming that sources vary sinusoidally sometimes makes the solution of Maxwell's equations easier. Also, the responses to systems having sources that vary arbitrarily with time can be determined by superposition of the responses of the system due to sources varying with individual sinusoidal frequency components.

Consider the general second order system

$$\frac{d^2r}{dt^2} + a\frac{dr}{dt} + br = f$$

with *forcing function* $f = f_o\cos(\omega t + \theta_f)$. It is often easier to represent the forcing function as

$$f = Re\{f_o e^{j\theta_f} e^{j\omega t}\} = Re\{Fe^{j\omega t}\}$$

where $F = f_o e^{j\theta_f}$ is called a *complex phasor*. Because the system is linear

$$r = r_o\cos(\omega t + \theta_r)$$

or

$$r = Re\{Re^{j\omega t}\}$$

where $R = r_o e^{j\theta_r}$. Substitution of these into the general system expression yields

$$(b + aj\omega + -\omega^2)R = F$$

where the common time-dependent factor $e^{j\omega t}$ has been dropped. It is apparent that the so called *time-harmonic* system equation can be readily solved for the unknown R, and the original system variable r may be recovered from

$$r = Re\{Re^{j\omega t}\} .$$

In a similar way, a time-harmonic electric field may be represented

$$\vec{E}(x,y,z,t) = Re\left\{\vec{E}(x,y,z)e^{j\omega t}\right\} .$$

This results in a system of Maxwell's equations

$$\nabla\times\vec{E} = -j\omega\mu\vec{H}$$

$$\nabla\times\vec{H} = \vec{J} + j\omega\varepsilon\vec{E}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

$$\nabla \cdot \vec{H} = 0 .$$

2.5 Electromagnetic basis of Kirchoff's voltage and current laws

Although Maxwell's equations describe all electromagnetic phenomena, direct application of these laws is not always convenient. Often, approximations are employed to provide an easier method of analysis. Kirchoff's voltage and current laws are idealizations of Maxwell's laws which form the foundation of electric circuit theory. These laws remain valid when the elements of the circuit under analysis are small compared to a wavelength (the circuit is operated at a low frequency). The fields within the elements are then said to be *quasi-static*. It is also assumed that the circuit is composed of lumped elements. With certain types of circuits, such as transmission lines, effects due to distributed parameters must be taken into account (this will be addressed in Section 2.6). The pages that follow will illustrate the relationships between Kirchoff's laws and Maxwell's equations.

- **relationship between KVL and Faraday's Law**

Consider a generalized electric circuit consisting of a source and a resistive element which are connected by perfectly conducting wires.

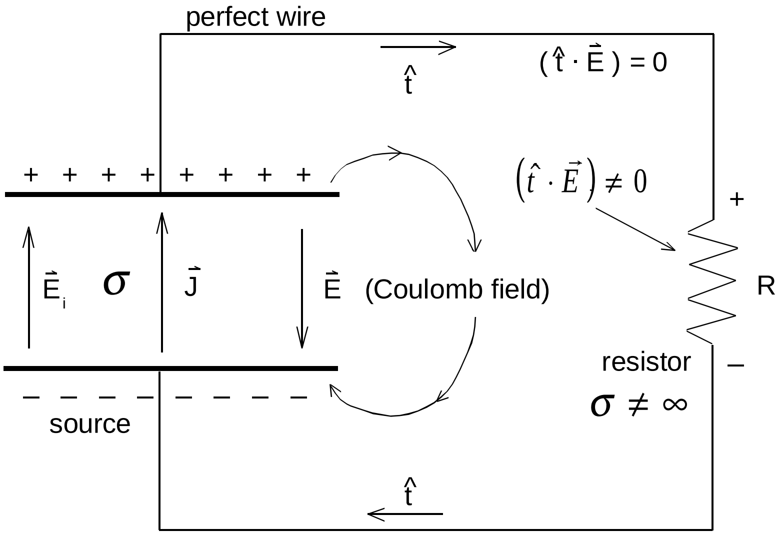

Figure 7.Generalized electric circuit.

An impressed field $\vec{E}_i$ drives a current density $\vec{J}$ through the source region against the action of a Coulomb field $\vec{E}$. At any point in the circuit

$$\vec{J} = \sigma(\vec{E} + \vec{E}_i)$$

where $\vec{E}_i \neq 0$ only in the source region. Consequently

$$\oint_C (\vec{E} + \vec{E}_i)\cdot d\vec{l} = \oint_C \frac{\vec{J}}{\sigma}\cdot d\vec{l}$$

or

$$\oint_C \vec{E}\cdot d\vec{l} + \int_{source} \vec{E}_i\cdot d\vec{l} = I\left[\int_{source} \frac{dl}{\sigma S} + \int_{element} \frac{dl}{\sigma S}\right].$$

For a static electric field

$$\oint_C \vec{E}\cdot d\vec{l} = 0$$

and, applying this, the above result leads to

$$V = I\,(R_{source} + R)$$

where

$$V = \int_{source} \vec{E}_i\cdot d\vec{l}$$

is the impressed emf. When multiple source and resistive regions exist in the circuit, this generalizes to

$$\sum_j V_j = \sum_k I_k R_k$$

which states that the sum of the emfs is equal to the sum of the voltage drops around a closed circuit which supports steady currents (Kirchoff's voltage law). For time-dependent currents, Faraday's law requires

$$\oint_C \vec{E}\cdot d\vec{l} \;=\; -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S} \;=\; -\frac{d}{dt}\Big[L\,i(t)\Big]$$

where the latter term follows from the definition of inductance L. This leads to the time-dependent generalization

$$v(t) \;=\; \Big(R_{source} + R\Big)\, i(t) + L\,\frac{d\,i(t)}{dt}\,.$$

- **relationship between KCL and the continuity equation**

The continuity equation requires

$$\nabla\cdot\vec{J} \;=\; -\frac{\partial\rho}{\partial t}\,.$$

For steady currents $\partial\rho/\partial t \;=\; 0$, therefore

$$\nabla\cdot\vec{J} \;=\; 0\,.$$

This leads to the integral form

$$\oint_S \vec{J}\cdot d\vec{s} \;=\; 0$$

which may be written

$$\sum_j I_j \;=\; 0\,.$$

This states that the sum of the currents flowing out of a circuit node is zero (Kirchoff's current law). It is important to note that KCL is rigorously valid only when there is no time-dependent charge accumulation at nodes in the circuit, otherwise that charge must be modeled.

2.6 Transmission lines

Unlike dc or low-frequency (60 Hz) circuits, transmission lines may be many wavelengths long. Where the various elements of a low-frequency circuit may be considered to be discrete (lumped), transmission lines must be described in terms of parameters which are distributed throughout their entire length.

- **general transmission line equations (time domain)**

Consider a short element of transmission line with length Δz. This section of line, can be described in terms of per-unit-length distributed parameters, consisting of a series resistance R, a series inductance L, a shunt capacitance C, and a shunt conductance G. Application of Kirchoff's voltage law to this element yields

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}$$

and application of Kirchoff's current law yields

$$\frac{\partial i}{\partial z} = -Gv - C\frac{\partial v}{\partial t} .$$

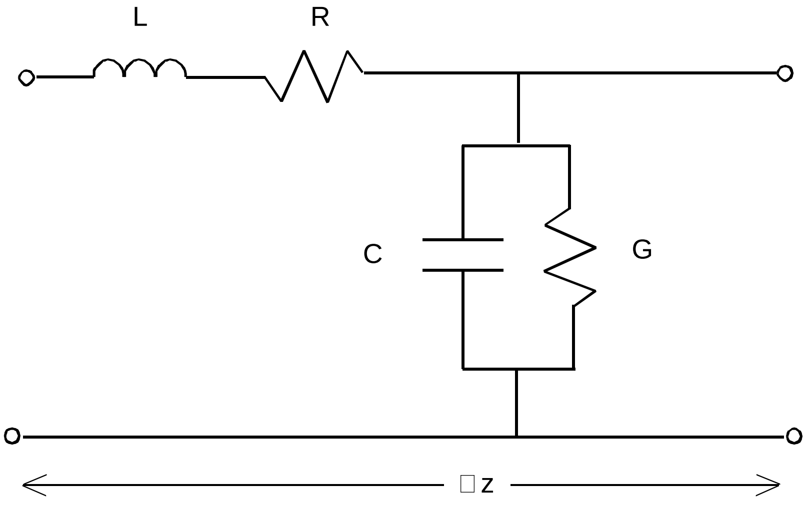

Figure 8.General transmission line element.

These may be represented for time-harmonic quantities as

$$\frac{dV(z)}{dz} = -ZI(z)$$

where

$$Z = (R + j\omega L)$$

and

$$\frac{dI(z)}{dz} = -YV(z)$$

with

$$Y = (G + j\omega C)\,.$$

These coupled time-harmonic expressions can be solved simultaneously to determine $V(z)$ and $I(z)$ yielding

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0$$

and

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

where

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{ZY}$$

is known as the *propagation constant*, having real part α, the *attenuation constant*, and imaginary part β, the *phase constant*. It can be seen that the expressions for $V(z)$ and $I(z)$ above have solutions

$$V(z) = V^+e^{-\gamma z} + V^-e^{\gamma z}$$

and

$$I(z) = -\frac{1}{Z}\frac{dV}{dz} = \frac{1}{Z_o}(V^{+}e^{-\gamma z} - V^{-}e^{\gamma z})$$

where V^{+} and V^{-} are the constant amplitudes of forward and backward traveling waves, respectively, and

$$Z_o = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

is known as the *characteristic impedance* of the transmission line. It should be noted that in sinusoidal steady state, all of the multiply reflected forward and backward traveling waves excited superpose resulting in a single forward traveling wave of amplitude V^{+}, and a single backward traveling wave of amplitude V^{-}. These waves interfere to produce a standing wave interference pattern along the line.

A lossless transmission line is characterized by R=G=0. In this case the impedance of the transmission line is purely real where

$$Z_o = R_o = \sqrt{\frac{L}{C}}$$

is referred to as the *characteristic resistance* of the transmission line.

- **phase velocity and wavelength**

In sinusoidal steady-state, the voltage and current along a transmission line are represented by

$$V(z,t) = V(z)\,e^{j\omega t} = V^{+}e^{-\alpha z}e^{j\omega(t - \beta z/\omega)} + V^{-}e^{\alpha z}e^{j\omega(t + \beta z/\omega)}$$

and

$$I(z,t) = I(z)\,e^{j\omega t} = \frac{1}{Z_o}\left[V^{+}e^{-\alpha z}e^{j\omega(t - \beta z/\omega)} - V^{-}e^{\alpha z}e^{j\omega(t + \beta z/\omega)}\right].$$

Here, the term

$$e^{j\omega(t \mp \beta z/\omega)}$$

describes the phase of the waves; therefore $(t \mp \beta z/\omega)$ is a constant at constant phase points of the traveling waves. Therefore

$$\frac{d}{dt}(t \mp \beta z/\omega) = \frac{d}{dt}(const.)$$

which indicates

$$1 \mp \frac{\beta}{\omega}\frac{dz}{dt} = 0\,.$$

Here

$$v_p = \frac{dz}{dt} = \pm\frac{\omega}{\beta}$$

is known as *phase velocity*, and represents the velocity with which a reference point of constant phase of the wave advances (propagates) along the transmission line. It is now clear that the expression

$$V^{\pm}e^{\mp\alpha z}e^{j\omega(t \mp \beta z/\omega)} = V^{\pm}e^{\mp\alpha z}e^{j\omega(t \mp z/v_p)}$$

represents waves traveling in the $\pm$z direction with phase velocities $\pm v_p = \pm\omega/\beta$ and attenuation constant α.

It was noted previously that the distance λ between any two adjacent equiphase points of a single traveling wave (at an instant t) is the wavelength of that wave. If equiphase points occur at locations z and Δz on a transmission line, then

$$e^{j(\omega t \mp \beta z)} = e^{j[\omega t \mp \beta(z + \Delta z)]}$$

which means that

$$e^{\mp j\beta\Delta z} = 1\,.$$

From this, it is seen that

$$\beta\Delta z = n2\pi$$

where n is any non-zero, positive integer (n=1,2,3,...). The spacing between equiphase points is then

$$\Delta z = n\frac{2\pi}{\beta} = n\lambda$$

where

$$\lambda = \frac{2\pi}{\beta}$$

is the wavelength, or distance between adjacent equiphase points along a transmission line. The relationship between wavelength and phase velocity is given by

$$\lambda = \frac{2\pi}{\beta} = 2\pi\frac{v_p}{\omega} = \frac{2\pi v_p}{2\pi f} = \frac{v_p}{f}.$$

- **input impedance, reflection and transmission coefficients**

When a transmission line of finite length is terminated with a load impedance that does not match the line, reflections occur. Thus at any point along the line both backward and forward traveling waves exist. For a transmission line of characteristic impedance Z_o, with length l, having propagation constant γ, the voltage and current at any point along the line are given by

$$V(z) = V^+\left(e^{-\gamma z} + \frac{V^-}{V^+}e^{\gamma z}\right) = V^+\left(e^{-\gamma z} + \Gamma e^{\gamma z}\right)$$

$$I(z) = \frac{V^+}{Z_o}\left(e^{-\gamma z} - \frac{V^-}{V^+}e^{\gamma z}\right) = \frac{V^+}{Z_o}\left(e^{-\gamma z} - \Gamma e^{\gamma z}\right)$$

where again V^+ and V^- are the forward and backward traveling voltage wave amplitudes, respectively, and

$$\Gamma = \frac{V^-}{V^+}$$

is the *reflection coefficient* at the receiving end terminals. Now, at the receiving (load) end

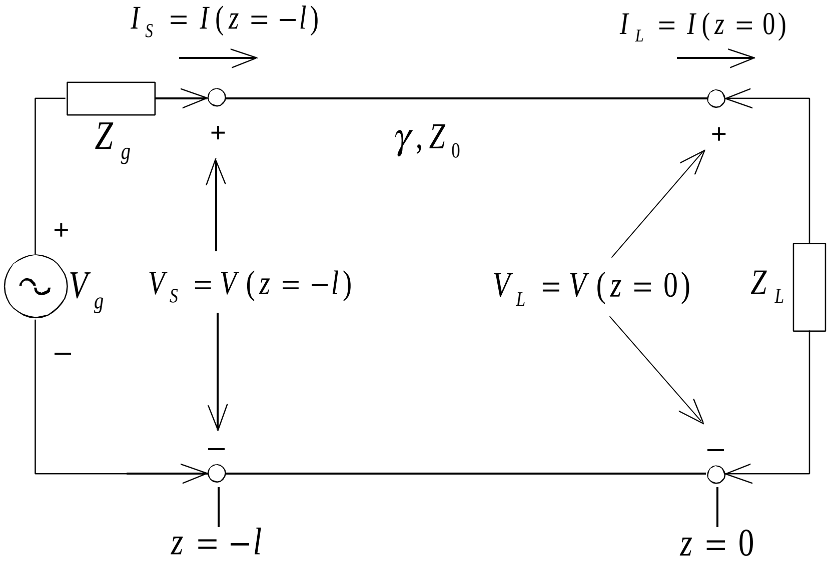

Figure 9.Finite-length transmission line with load impedance.

of the transmission line (z=0), the impedance boundary condition gives

$$V(z{=}0) \ = \ Z_L I(z{=}0)$$

which leads to

$$V^+(1+\Gamma) \ = \ \frac{Z_L V^+}{Z_o}(1-\Gamma) \, .$$

Solving this expression for Γ gives

$$\Gamma \ = \ \frac{V^-}{V^+} \ = \ \frac{Z_L - Z_o}{Z_L + Z_o}$$

which is the receiving end reflection coefficient. Since

$$V(\mathrm{z}=0) \ = \ V^+(1+\Gamma) \ = \ TV^+$$

the receiving end *transmission coefficient* is given by

$$T = (1 + \Gamma) = \frac{2Z_L}{Z_L + Z_o} .$$

The impedance at any point along the line is

$$Z(z) = \frac{V(z)}{I(z)} = Z_o\left(\frac{e^{-\gamma z} + \Gamma e^{\gamma z}}{e^{-\gamma z} - \Gamma e^{\gamma z}} \right) .$$

Substituting the expression for Γ above gives

$$Z(z) = Z_o\left[\frac{(Z_L + Z_o) e^{-\gamma z} + (Z_L - Z_o) e^{\gamma z}}{(Z_L + Z_o) e^{-\gamma z} - (Z_L - Z_o) e^{\gamma z}} \right].$$

Applying the relationships

$$\cosh\gamma z = \frac{e^{\gamma z} + e^{-\gamma z}}{2}, \qquad \sinh\gamma z = \frac{e^{\gamma z} - e^{-\gamma z}}{2}$$

the expression for impedance becomes

$$Z(z) = Z_o\left(\frac{Z_L \cosh\gamma z - Z_o \sinh\gamma z}{Z_o \cosh\gamma z - Z_L \sinh\gamma z} \right)$$

or

$$Z(z) = Z_o \frac{Z_L - jZ_o \tanh\gamma z}{Z_o - jZ_L \tanh\gamma z} .$$

At the sending end of the transmission line (z=-l), this becomes

$$Z_i = Z(z = -l) = Z_o \frac{Z_L + jZ_o \tanh\gamma l}{Z_o + jZ_L \tanh\gamma l} .$$

This expression represents the impedance seen at the input terminals of the transmission line and is therefore known as the *input impedance*.

- **impedance transformations**

Transmission lines are sometimes used as UHF circuit elements. At these frequencies (300 MHz - 3 GHz) sections of transmission line may be used to match a load to the internal impedance of a generator or to the characteristic impedance of another transmission line (stub matching). These segments of matching transmission line are usually considered to be lossless. In this case $\gamma = j\beta$, $Z_o=R_o$, and $\tanh\gamma l = \tanh(j\beta l) = j\tan\beta l$. The input impedance of a section of lossless transmission line is therefore

$$Z_i = R_o \frac{Z_L + jR_o \tan\beta l}{R_o + jZ_L \tan\beta l} .$$

The input impedance for several load terminations will be examined.

- open circuit termination

If the transmission line segment is open circuited, then $Z_L=\infty$, and

$$Z_i = -\frac{jR_o}{\tan\beta l} = -jR_o \cot\beta l .$$

It is seen from this that the input impedance of an open circuited section of lossless transmission line is purely reactive, regardless of the length of the line. This reactance may be inductive or capacitive, depending on the sign of $\cot\beta l$.

- short circuit termination

If the transmission line segment is short circuited, then $Z_L=0$, and

$$Z_i = jR_o \tan\beta l .$$

It is seen from this that the input impedance of a short-circuited section of lossless transmission line is also purely reactive, regardless of the length of the line. Again, this reactance may be inductive or capacitive, depending on the sign of $\tan\beta l$.

Important note: From the information presented above, it can be seen that

$$(Z_i)_{short}(Z_i)_{open} = Z_o \tanh(\gamma l)\, Z_o \coth(\gamma l) = Z_o^2 .$$

Thus the characteristic impedance of an unknown transmission line can be determined by making measurements with short- and open-circuit terminations

$$Z_o = \sqrt{(Z_i)_{short}(Z_i)_{open}}\,.$$

This applies regardless of whether the transmission line is lossy or lossless.

- **half-wavelength line**

When the length of a transmission line is an integer multiple of one-half of a wavelength ($l = n\lambda/2$ for n=1,2,3,...), then

$$\beta l = \frac{2\pi}{\lambda}\left(\frac{n\lambda}{2}\right) = n\pi$$

which makes

$$\tan\beta l = 0$$

and the input impedance becomes

$$Z_i = Z_L\,.$$

Thus for a lossless line which is some multiple of a half-wavelength long, the input impedance is the same as the load impedance.

- **quarter-wavelength line**

When the length of a transmission line is an odd integer multiple of one-quarter of a wavelength ($l = (2n-1)\lambda/4$ for n=1,2,3,...), then

$$\beta l = \frac{2\pi}{\lambda}(2n-1) = (2n-1)\frac{\pi}{2}$$

which makes

$$\tan\beta l = \tan\left[(2n-1)\frac{\pi}{2}\right] \rightarrow \pm\infty$$

and the input impedance becomes

$$Z_i = \frac{R_o^2}{Z_L}.$$

Thus for a lossless line which is some multiple of a quarter-wavelength long, the input impedance is proportional to the inverse of the load impedance. A quarter wavelength transmission line is sometimes referred to as a *quarter-wave transformer*.

- **voltage standing wave ratio (VSWR)**

When transmission lines are excited by sinusoidal, continuous wave (CW) sources, forward traveling voltage and current waves propagate away from the source, and backward traveling waves are reflected from discontinuities and mismatched terminations. These forward and backward traveling waves interfere to form a standing wave pattern on the line. The ratio of the maximum to the minimum voltage of the standing wave is known as the *voltage standing wave ratio*. Since the voltage at any point along the transmission line is given by

$$V(z) = V^+ e^{-j\beta z}(1 + \Gamma e^{j2\beta z})$$

the standing wave ratio is therefore

$$S = \frac{|V_{\max}|}{|V_{\min}|} = \frac{1+|\Gamma|}{1-|\Gamma|}.$$

The VSWR is a measure of the nature and magnitude of mismatches or terminations that occur on the transmission line. A VSWR of 1 results when $|\Gamma| = 0$, indicating that the transmission line is match-terminated, and no backward traveling wave exists. A VSWR of infinity indicates that the signal is being totally reflected by either an open-circuit, short-circuit, or purely reactive termination.

2.7 Network theory

- **two port networks, S-parameters, Z-parameters, Y-parameters**

The study of two port networks is important in the field of electrical engineering because most electric circuits and electronic modules have at least two ports, namely input and output terminal pairs. Two-port parameters describe a system in terms of the voltage and current that may be measured at each port. A typical generalized two-port network is indicated in the figure below.

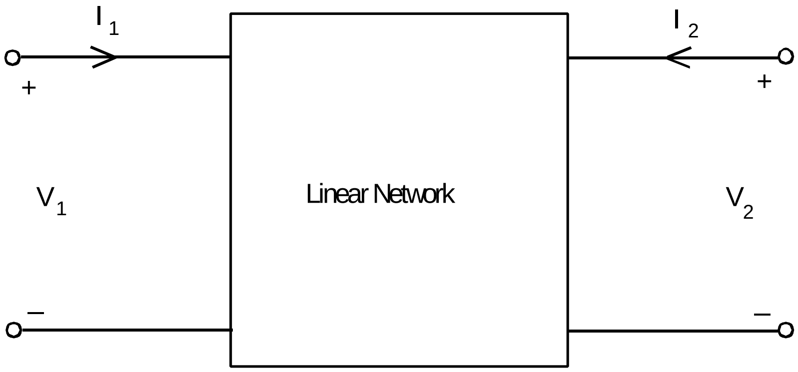

Figure 10.General linear network.

Here I_1 is the current entering port 1, I_2 is the current entering port 2, and V_1 and V_2 are voltages that exist at ports 1 and 2, respectively.

- **Y-parameters**

If the network that exists between ports 1 and 2 is *linear*, and contains no independent sources, then the principle of superposition may be applied to determine the currents I_1 and I_2 in terms of voltages V_1 and V_2

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2 .$$

These may be written in matrix form as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.$$

Here the terms Y_{11}, Y_{12}, Y_{21}, and Y_{22} are known as *admittance*, or *Y-parameters*. It is apparent form the matrix equation above that the parameter Y_{11} may be determined by measuring I_1 and V_1 when V_2 is equal to zero (or more accurately when port 2 is short-circuited)

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0}.$$

The remaining Y-parameters may be determined in a similar manner

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1 = 0}$$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2 = 0}$$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0}.$$

Because the parameter Y_{11} is an admittance which is measured at the input of a two-port network when port 2 is short circuited, it is known as the *short-circuit input admittance*. Likewise, the parameters Y_{12} and Y_{21} are known as *short-circuit transfer admittances*, and Y_{22} is the *short-circuit* output admittance. It is evident that, using these relationships, the properties of an unknown, linear two-port network can be completely specified by values measured experimentally at ports 1 and 2.

- **Z-parameters**

A similar set of relationships may be established in which the voltages at the input and output of a linear two-port network are expressed in terms of the input and output currents

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

which may be described in matrix form

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} .$$

It is apparent from this expression that the parameter Z_{11} may be determined by open-circuiting port 2 (letting I_2 equal zero)

$$Z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2 = 0}$$

and the remaining *impedance* or *Z-parameters* are determined similarly as

$$Z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1 = 0}$$

$$Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2 = 0}$$

$$Z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1 = 0} .$$

The term Z_{11} is known as the *open-circuit input impedance*, Z_{12}, and Z_{21} are known as *open-circuit transfer impedances*, and Z_{22} is the *open-circuit output impedance*.

- **S-parameters**

The representations above are useful if voltage and current can easily be measured at the input and output of the two-port network. While it is usually possible to directly measure both voltage and current in low frequency electric circuits, it is not always possible to do this with high-frequency circuits or particularly with waveguides. In such cases it is often necessary to determine impedances from measured standing wave ratios or reflection coefficients. It is thus convenient to describe unknown high-frequency networks in terms of outgoing and incoming wave amplitudes, instead of voltages and currents. The figure below represents a linear two-port network

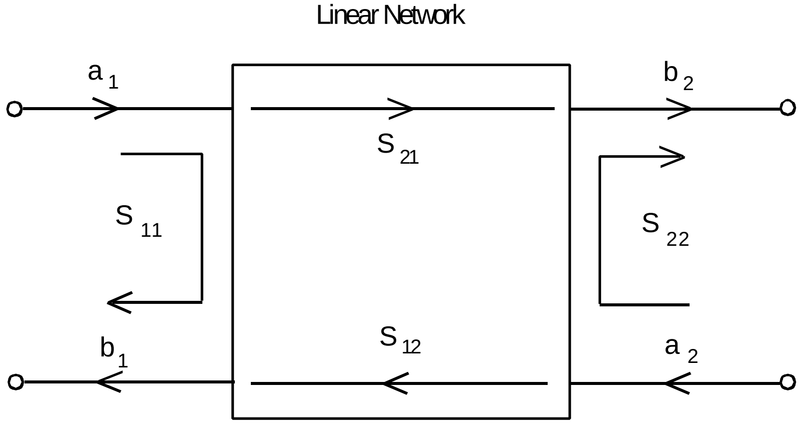

Figure 11.General scattering network.

where a_1, and a_2 are incoming wave amplitudes, and b_1, and b_2 are outgoing wave amplitudes. These parameters represent the normalized incident and reflected voltage wave amplitudes. For an n-port network they are

$$a_n = \frac{V_n^+}{\sqrt{Z_{on}}}, \qquad b_n = \frac{V_n^-}{\sqrt{Z_{on}}}$$

where Z_{on} is the equivalent characteristic impedance at the nth terminal port. The voltage and current at some reference plane within the network can thus be represented by

$$V_n = V_n^+ + V_n^- = \sqrt{Z_{on}}(a_n + b_n)$$

and

$$I_n = \frac{1}{Z_{on}}(V_n^+ - V_n^-) = \frac{1}{\sqrt{Z_{on}}}(a_n - b_n) .$$

Solving for a_n and b_n yields

$$a_n = \frac{1}{2}\left(\frac{V_n}{\sqrt{Z_{on}}} + \sqrt{Z_{on}}\, I_n \right)$$

$$b_n = \frac{1}{2}\left(\frac{V_n}{\sqrt{Z_{on}}} - \sqrt{Z_{on}}\, I_n \right) .$$

The average power flowing into the nth terminal is given by

$$(P_n)_{ave} = \frac{1}{2} Re(V_n I_n^*) = \frac{1}{2} Re\left[(a_n a_n^* - b_n b_n^*) + (b_n a_n^* - b_n^* a_n) \right] .$$

The term $(a_n a_n^* - b_n b_n^*)$ is purely real, and the term $(b_n a_n^* - b_n^* a_n)$ is purely imaginary, therefore

$$(P_n)_{ave} = \frac{1}{2}(a_n a_n^* - b_n b_n^*)$$

where $(P_n)_{ave}^{+} = 1/2\,(a_n a_n^*)$ represents power flow into the terminal, and $(P_n)_{ave}^{-} = 1/2\,(b_n b_n^*)$ represents the power reflected out of the terminal.

Consider again the case of a network with two terminal ports. The principle of superposition may be applied to represent the outgoing waves in terms of a linear combination of the incoming waves

$$b_1 = S_{11} a_1 + S_{12} a_2$$

$$b_2 = S_{21} a_1 + S_{22} a_2$$

which may be written in matrix form as

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} .$$

The individual *scattering*, or *S-parameters* are determined in a way similar to those above. The parameter S_{11} is the ratio of the outgoing wave amplitude at port 1 to the incoming wave amplitude at port 1 when the incoming wave amplitude at port 2 is zero (or more correctly when port 2 is match terminated)

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} .$$

It is noted that the ratio of the outgoing wave amplitude at port 1 to the incoming wave amplitude at port 1 is simply the reflection coefficient at port 1. Therefore S_{11} is the reflection coefficient at port 1 when port 2 is match terminated. The remaining S-parameters are determined as follows

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0}$$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} .$$

2.8 Electromagnetic radiation

Electromagnetic radiation is the transfer of EM energy from electric sources to EM waves in space. Under certain conditions it is sufficient to examine only the propagation characteristics of waves and the interactions between fields. In these so-called *source free regions,* it is not necessary to know anything about how the source of EM energy behaves. Waves simply exist in a medium, and how they came to be there is not important. Under other conditions, however, it *is* necessary to understand the relationship between fields and time varying source charge and current distributions. Such is the case with antenna theory.

- **uniform plane waves in lossless media**

A plane electromagnetic wave is one for which points of constant amplitude and phase are contained in planar surfaces (wavefronts). Localized radiating electric sources, such as those associated with antennas, actually produce spherical waves. However, at distances far from the source, these disturbances may be locally approximated as being planar.

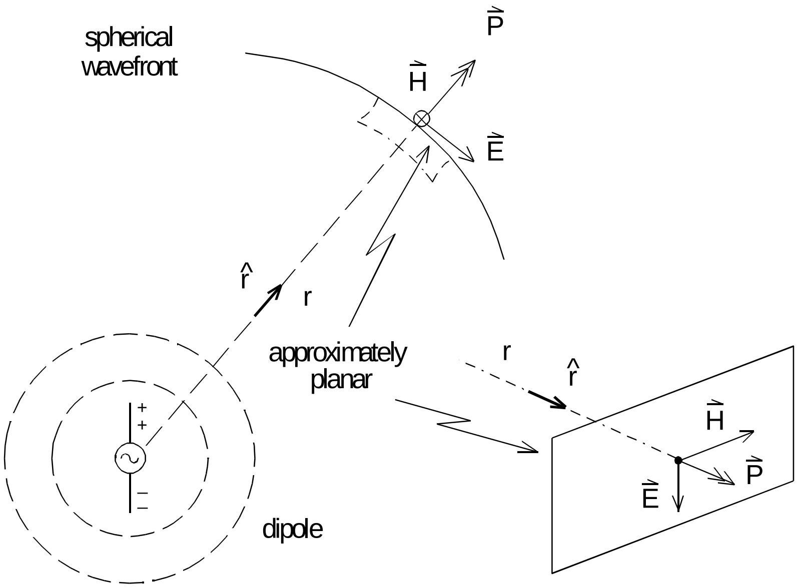

Figure 12.Planar approximation of spherical wavefront.

- **plane-wave solutions to homogeneous Helmholtz equation**

Previously, the homogeneous Helmholtz equation for the electric field in a source free region was found to be

$$\nabla^2\vec{E} + k^2\vec{E} = 0 .$$

The general vector solution to this equation

$$\vec{E}(\vec{r}) = \vec{E}_o e^{-j\vec{k}\cdot\vec{r}}$$

is found using the method of separation of variables. Here $\vec{E}_o$ is a constant vector amplitude, $\vec{r} = \hat{x}x + \hat{y}y + \hat{z}z$ is a position vector, and $k^2 = \omega^2\mu\varepsilon_c$. The wavenumber is $k = \beta - j\alpha$ and the complex permittivity is $\varepsilon_c = (1 - j\sigma/\omega\varepsilon)$. This expression for $\vec{E}(\vec{r})$ has the following physical interpretation:

i. $\vec{E}(\vec{r})$ is a wave propagating in the direction of $\vec{k} = \hat{k}k$. The scalar components of $\vec{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$ are phase constants for propagation along the x, y, z directions.

ii. Points of constant amplitude and phase of the wave are located on a planar surface which is perpendicular to $\vec{k}$, which points in the direction of propagation of the plane wave.

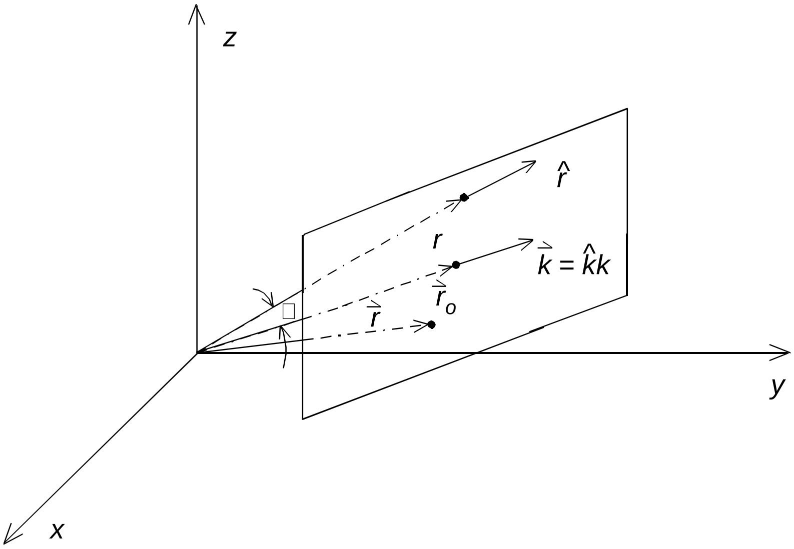

Figure 13.Planar wavefront.

The instantaneous complex electric field is given by

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})e^{j\omega t} = \vec{E}_o e^{j(\omega t - \vec{k}\cdot\vec{r})} = \vec{E}_0 e^{-\alpha\hat{k}\cdot\vec{r}} e^{(\omega t - \beta\hat{k}\cdot\vec{r})}$$

where

$$e^{j(\omega t - \beta\hat{k}\cdot\vec{r})}$$

is a constant by the criterion for constant phase of the wave. This implies that

$$\beta\hat{k}\cdot\vec{r} = \beta r(\hat{k}\cdot\hat{r}) = \beta r\cos\theta = \beta r_o = constant$$

which requires that $\vec{r}$ be located on a plane surface which is perpendicular to $\vec{k}$.

- **relationship between electric and magnetic fields for a plane wave**

The time-harmonic form of Faraday's law in a source free region states

$$\nabla\times\vec{E} = -j\omega\mu\vec{H}.$$

Substitution of the plane wave electric field representation into this expression gives

$$\vec{H} = \frac{j}{\omega\mu}\nabla\times\vec{E} = \frac{j}{\omega\mu}\nabla\times\left(\vec{E}_o e^{-j\vec{k}\cdot\vec{r}}\right)$$

Application of the vector identity

$$\nabla\times(\psi\vec{A}) = \psi\nabla\times\vec{A} + \nabla\psi\times\vec{A}$$

leads to

$$\vec{H} = \frac{j}{\omega\mu}\left[e^{-j\vec{k}\cdot\vec{r}}\nabla\times\vec{E}_o + \nabla\left(e^{-j\vec{k}\cdot\vec{r}}\right)\times\vec{E}_o\right].$$

Now

$$\nabla\times\vec{E}_o = 0$$

because the curl of a constant vector is zero, therefore

$$\vec{H} = \frac{j}{\omega\mu}\left(-j\vec{k}\times\vec{E}_o\right)e^{-j\vec{k}\cdot\vec{r}} = \frac{k}{\omega\mu}\left(\hat{k}\times\vec{E}_o\right)e^{-j\vec{k}\cdot\vec{r}}.$$

The *intrinsic impedance* or *wave impedance* of the simple medium is defined as

$$\eta = \frac{\omega\mu}{k}$$

thus

$$\vec{H} = \frac{\hat{k}\times\vec{E}_o}{\eta}e^{-j\vec{k}\cdot\vec{r}} = \frac{\hat{k}\times\vec{E}}{\eta}.$$

- **phase velocity, group velocity, and wavelength**

Phase velocity represents the speed that an imaginary observer would have to travel along a direction $\hat{r}$ in order to remain adjacent to a reference point of constant phase along a wave front, i.e.,

$$e^{j(\omega t - \beta(\hat{k}\cdot\vec{r}))} = constant$$

which implies

$$\omega t - \beta(\hat{k}\cdot\vec{r}) = \omega t - \beta r(\hat{k}\cdot\hat{r}) = constant\,.$$

The time derivative of a constant is zero

$$\frac{d}{dt}\left[\omega t - \beta r(\hat{k}\cdot\hat{r})\right] = \frac{d}{dt}(constant) = 0$$

therefore

$$\omega - \beta(\hat{k}\cdot\hat{r})\frac{dr}{dt} = 0\,.$$

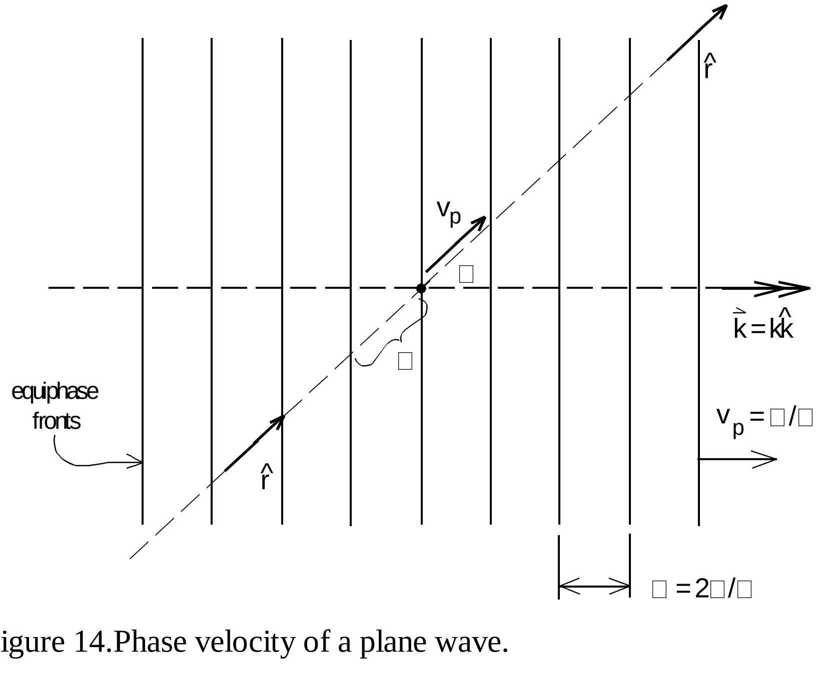

Figure 14.Phase velocity of a plane wave.

From this, the phase velocity along a direction $\hat{r}$ is found to be

$$v_p = \frac{dr}{dt} = \frac{\omega}{\beta(\hat{k}\cdot\hat{r})} = \frac{\omega}{\beta\cos\theta}$$

where θ is the angle between the direction of wave propagation ($\hat{k}$), and the direction along which the observer moves ($\hat{r}$). The minimum phase velocity $v_p = \omega/\beta$ occurs when the observer moves parallel to $\vec{k}$ such that $(\hat{r}\cdot\hat{k}) = 1$ $(\theta = 0)$, i.e., when the observer moves in the direction of propagation of the wave. It is assumed that $\theta = 0$ unless otherwise specified.

In a lossless medium, the propagation constant $\beta = \omega\sqrt{\mu\varepsilon}$ depends linearly upon frequency. In some cases, however, the propagation constant is not a linear function of frequency. In these cases waves of different frequencies will propagate with different phase velocities, which leads to a type of signal distortion known as dispersion. A signal containing a small spread, or "group" of frequency components will tend to "spread out", or disperse, as it propagates. A *group velocity* may be defined, which is the velocity of propagation of this "frequency packet"

$$v_g = \frac{1}{d\beta/d\omega}\,.$$

The wavelength λ of a plane EM wave along a direction $\hat{r}$ is the distance between any two adjacent points r_1 and r_2 with identical phase at any instant t, i.e.,

$$e^{j[\omega t - \beta(\hat{k}\cdot\vec{r}_1)]} = e^{j[\omega t - \beta(\hat{k}\cdot\vec{r}_2)]}$$

therefore

$$e^{-j\beta\hat{k}\cdot(\vec{r}_2 - \vec{r}_1)} = e^{-j\beta(\hat{k}\cdot\hat{r})(r_2 - r_1)} = 1$$

because $\vec{r}_1 = \hat{r}r_1$, $\vec{r}_2 = \hat{r}r_2$, and thus

$$\beta(r_2 - r_1)(\hat{k}\cdot\hat{r}) = 2\pi$$

for adjacent equiphase points. The wavelength along a direction $\hat{r}$ is now seen to be

$$\lambda = (r_2 - r_1) = \frac{2\pi}{\beta(\hat{k}\cdot\hat{r})} = \frac{2\pi}{\beta\cos\theta}$$

where again $\theta = \cos^{-1}(\hat{k}\cdot\hat{r})$. Now, from the definition of phase velocity

$$\beta(\hat{k}\cdot\hat{r}) = \frac{\omega}{v_p}$$

which leads to an expression of wavelength which is independent of θ

$$\lambda = \frac{2\pi}{\beta(\hat{k}\cdot\hat{r})} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}.$$

- **power flow**

Previously the electric and magnetic fields associated with a plane EM wave were found to be

$$\vec{E} = \vec{E}_o e^{-j\vec{k}\cdot\vec{r}} = \vec{E}_o e^{-j(\beta - j\alpha)(\hat{k}\cdot\vec{r})}$$

and

$$\vec{H} = \frac{(\hat{k}\times\vec{E})}{\eta}.$$

The time-average power flow associated with a plane EM wave is described by Poynting's power density, which is given by

$$\vec{P} = \frac{1}{2}Re\left\{\vec{E}\times\vec{H}^*\right\}.$$

Substitution of the above expressions for the electric and magnetic fields leads to

$$\vec{P} = \frac{1}{2}Re\left\{\vec{E}\times\left(\frac{\hat{k}\times\vec{E}}{\eta}\right)^*\right\}$$

$$= \frac{1}{2}Re\left\{\frac{1}{\eta^*}\left[(\vec{E}\cdot\vec{E}^*)\hat{k} - (\vec{E}\cdot\hat{k})\vec{E}^*\right]\right\}$$

$$= \hat{k}\frac{1}{2}\,Re\left\{\frac{\vec{E}\cdot\vec{E}^{\,*}}{\eta^*}\right\} = \hat{k}\frac{1}{2}\,Re\left\{\frac{\vec{E}_o\cdot\vec{E}_o^{\,*}}{\eta^*}e^{-2\alpha(\hat{k}\cdot\vec{r})}\right\}.$$

Thus the time-average power density of a plane EM wave is given by

$$\vec{P} = \hat{k}\,Re\left\{\frac{|\vec{E}_o|^2}{2\eta^*}\right\}e^{-2\alpha(\hat{k}\cdot\vec{r})}.$$

It should be noted that power flow is in the direction of wave propagation ($\hat{k}$). The exponential attenuation is due to progressive power dissipation in a lossy EM medium. For the special case of waves propagating in a lossless medium (e.g., free space) the attenuation constant α=0, and the intrinsic impedance of the medium η is real, therefore

$$\vec{P} = \hat{k}\frac{|\vec{E}_o|^2}{2\eta}.$$

- **one-dimensional plane waves**

For waves propagating in the $\pm z$ directions, $\hat{k} = \pm\hat{z}$ and $\vec{E}_o = \hat{x}E_o^{\pm}$, therefore

$$\vec{E}(z) = \hat{x}\left(E_o^+e^{-jkz} + E_o^-e^{jkz}\right)$$

and

$$\vec{H}(z) = \frac{1}{\eta}\left(\hat{z}\times\hat{x}E_o^+e^{-jkz} - \hat{z}\times\hat{x}E_o^-e^{jkz}\right)$$

$$= \frac{\hat{y}}{\eta}\left(E_o^+e^{-jkz} - E_o^-e^{jkz}\right).$$

Now a plane wave reflection coefficient may be defined such that

$$\Gamma = \frac{E_o^-}{E_o^+}$$

and the total one-dimensional plane wave field may then be expressed

$$\vec{E}(z) = \hat{x}\vec{E}_x(z) = \hat{x}E_o^+\left(e^{-jkz} + \Gamma e^{jkz}\right)$$

$$\vec{H}(z) = \hat{y}H_y(z) = \hat{y}\frac{E_o^+}{\eta}\left(e^{-jkz} - \Gamma e^{jkz}\right).$$

It should be noted that the fields $E_x(z)$ and $H_y(z)$ are exactly analogous, respectively, to V(z) and I(z) on a uniform transmission line. The concepts of standing waves, impedance $Z(z) = E_x(z)\big/H_y(z)$, standing wave ratio, and impedance transformation carry over directly.

- **waves in various media**

Remember that for a general material,

$$k = \omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} = \beta - j\alpha \quad .$$

i. waves in free space

In free space the wavenumber and attenuation constant are

$$k = \beta_o = \omega\sqrt{\mu_o\varepsilon_o}, \qquad \alpha = 0$$

the characteristic impedance of free space is given by

$$\eta = \eta_o = \omega\mu_o/\beta = \sqrt{\mu_o/\varepsilon_o} \approx 120\pi \ ohms$$

and the phase velocity of the waves is

$$v_p = c = \omega/\beta = 1/\sqrt{\mu_o\varepsilon_o} \approx 3\times10^8 m/s .$$

ii. waves in lossless media

In a lossless medium other than free space, the wavenumber and attenuation constant are

$$k = \beta = \omega\sqrt{\mu\varepsilon}, \qquad \alpha = 0$$

$$k = \beta_0\sqrt{\mu_r\varepsilon_r}$$

the impedance of the medium is

$$\eta = \omega\mu/k = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\mu_r\mu_o/\varepsilon_r\varepsilon_o} = \eta_o\sqrt{\mu_r\varepsilon_r}$$

and the phase velocity of the waves is

$$v_p = \omega/\beta = \frac{1}{\sqrt{\mu\varepsilon}} = 1/\sqrt{\mu_r\mu_o\varepsilon_r\varepsilon_o} = c/\sqrt{\mu_r\varepsilon_r}.$$

iii. waves in a good dielectric

In a good dielectric, $\sigma/\omega\varepsilon \ll 1$ for all frequencies of interest. In such a dielectric medium, the wavenumber is

$$k = \beta - j\alpha = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$

where

$$\varepsilon_c = \left(1 - j\frac{\sigma}{\omega\varepsilon}\right)$$

is known as the *complex equivalent permittivity* of the medium. According to the binomial expansion

$$(1 + x)^n \approx 1 + nx \qquad \ldots if\ x \ll 1$$

therefore

$$k \approx \omega\sqrt{\mu\varepsilon}\left(1 - j\frac{\sigma}{2\omega\varepsilon}\right).$$

The phase and attenuation constants are therefore given by

$$\beta \approx \omega\sqrt{\mu\varepsilon}$$

$$\alpha \approx \frac{\omega\sqrt{\mu\varepsilon}\sigma}{2\omega\varepsilon} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma\eta}{2}$$

the impedance of the medium is

$$\eta = \frac{\omega\mu}{\beta - j\alpha} = \frac{\omega\mu}{\beta}\frac{1}{1 - j\alpha/\beta} \approx \frac{\omega\mu}{\beta}\left(1 + j\frac{\alpha}{\beta}\right)$$

or

$$\eta \approx \omega\mu/\beta$$

because $\alpha/\beta \ll 1$, and the phase velocity of the waves is

$$v_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\varepsilon}}.$$

iv. waves in a good conductor

In a good conductor, $\sigma/\omega\varepsilon \gg 1$ for all frequencies of interest. Here, the wavenumber is

$$k = \beta - j\alpha = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} \approx \omega\sqrt{\mu\varepsilon}\sqrt{-j\frac{\sigma}{\omega\varepsilon}}$$

$$= \sqrt{\omega\mu\sigma}\left(e^{-j\pi/2}\right)^{1/2} = \sqrt{\omega\mu\sigma}\left(\frac{1 - j}{\sqrt{2}}\right) = (1 - j1)\sqrt{\frac{\omega\mu\sigma}{2}}$$

and it can be seen that the attenuation and phase constants are

$$\beta \approx \alpha = \sqrt{\pi f\mu\sigma}.$$

Now consider a plane EM wave with electric field

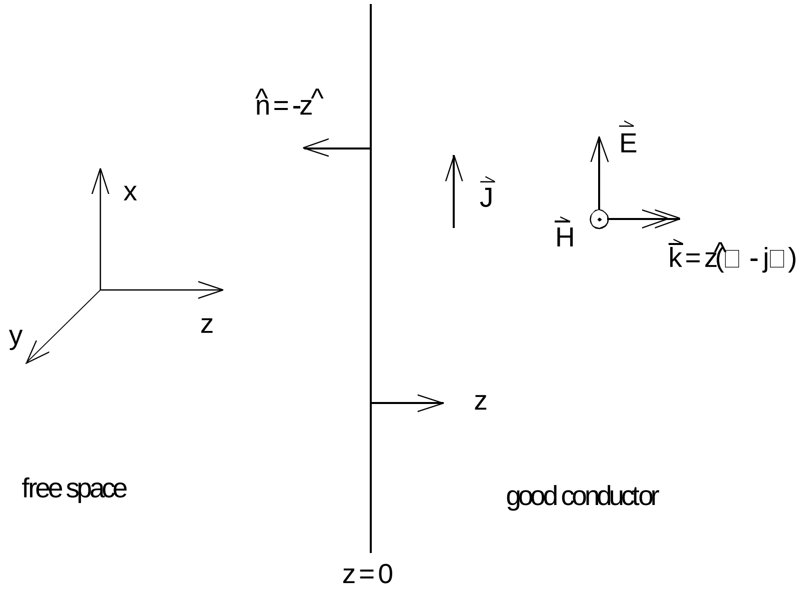

Figure 15.Plane EM waves in a good conductor.

$$\vec{E} = \vec{E}_o e^{j\vec{k}\cdot\vec{r}} = \hat{x}E_o e^{-j(\beta - j\alpha)(\hat{z}\cdot\vec{r})} = \hat{x}E_o e^{-\alpha z} e^{-j\beta z}$$

that exists within a good conductor. The associated current density is found using Ohm's law

$$\vec{J} = \sigma\vec{E} = \hat{x}\sigma E_o e^{-\alpha z} e^{-j\beta z} .$$

$\vec{E}$ and $\vec{J}$ decay to e^{-1} of their values E_o and σE_o at the conductor surface z=0 when $\alpha z=1$. The distance $z=\alpha^{-1}$ at this point is called the *skin depth* for waves in the conductor, and is given by

$$\delta = \alpha^{-1} = \frac{1}{\sqrt{\pi f \mu\sigma}} .$$

The impedance of the conducting medium is given by

$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]}} \approx (1 + j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1 + j)\frac{1}{\sigma\delta} .$$

The amount of current flowing per unit width in the region near the conductor surface is found from

$$\vec{K} = \int_0^1\int_0^\infty \hat{x}\, J_x(z)\, dz\, dy$$

$$= \hat{x}\sigma E_o \int_0^1 dy \int_0^\infty e^{-(\alpha + j\beta)z} dz = -\hat{x}\frac{\sigma E_o}{(\alpha + j\beta)} e^{-(\alpha + j\beta)z}\Bigg|_{z = 0}^{z = \infty}$$

$$= \hat{x}\frac{\sigma E_o}{(\alpha + j\beta)} = \hat{x}\frac{\sigma\delta E_o}{(1 + j1)} .$$

Using this result, the surface impedance is found by

$$z^i = \frac{E_x(z = 0)}{K_x} = \frac{E_o}{\sigma\delta E_o \big/ (1 + j1)} = \frac{(1 + j1)}{\sigma\delta}$$

or

$$z^i = r^i + j\omega l^i$$

where $r^i = \omega l^i = (\sigma\delta)^{-1}$ for a good conductor.

The magnetic field associated with the electric field presented above is found to be

$$\vec{H} = \frac{\hat{k}\times\vec{E}}{\eta} = \frac{\beta - j\alpha}{\omega\mu}\hat{z}\times\left(\hat{x}E_o e^{-\alpha z} e^{-j\beta z}\right)$$

$$= \hat{y}\frac{E_o(1 - j1)}{\omega\mu\delta} e^{-\alpha z} e^{-j\beta z} = \hat{y}\frac{2E_o}{\omega\mu\delta(1 + j1)} e^{-\alpha z} e^{-j\beta z}$$

$$= \hat{y}\frac{\sigma\delta E_o}{(1+j1)}e^{-\alpha z}e^{-j\beta z}\,.$$

From this it can be seen that

$$\hat{n}\times\vec{H}(z = 0) = -\hat{z}\times\vec{H}(z=0) = \hat{x}\frac{\sigma\delta E_o}{(1+j1)} = \vec{K}\,.$$

Thus the relationship between the magnetic field $\vec{H}$ present at the surface of a good conductor, and the surface current $\vec{K}$ is

$$\vec{K} = (\hat{n}\times\vec{H})\Big|_{surface}\,.$$

Thus it is seen that a current flowing on the surface of a good conductor will give rise to a magnetic field, which may be a source of interference elsewhere. Conversely, an external magnetic field may induce a current on a good conductor, which may in turn cause interference.

- **Maxwell's equations for EM field maintained by primary sources**

Consider a localized electric source distribution suspended in an unbounded medium.

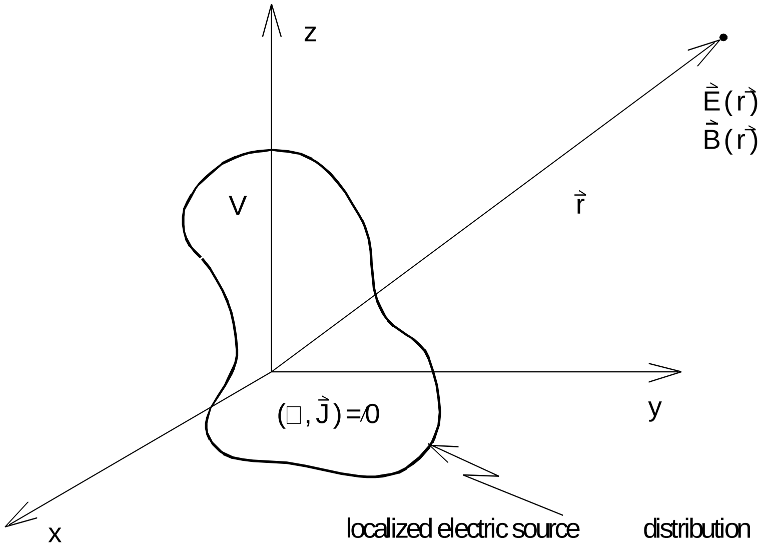

Figure 16.Localized source distribution.

The time-harmonic system of Maxwell's equations for this are

$$\nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_c}$$

$$\nabla \times \vec{E}(\vec{r}) = -j\omega \vec{B}(\vec{r})$$

$$\nabla \times \vec{B}(\vec{r}) = \mu \vec{J}(\vec{r}) + j\omega\mu\varepsilon_c \vec{E}(\vec{r})$$

$$\nabla \cdot \vec{B}(\vec{r}) = 0$$

where again $\varepsilon_c = \left[1 - j\sigma \big/ (\omega\varepsilon)\right]$ is the equivalent complex permittivity of the medium.

- **EM potential functions and Helmholtz equation**

It was shown that the electric and magnetic fields can be represented in terms of potential functions as

$$\vec{E} = -\nabla\Phi - j\omega\vec{A}$$

$$\vec{B} = \nabla\times\vec{A}.$$

Substitution of these expressions into Gauss's law leads to

$$\nabla\cdot\vec{E} = \nabla\cdot(-\nabla\Phi - j\omega\vec{A}) = \frac{\rho}{\varepsilon}$$

or

$$\nabla^2\Phi + j\omega\nabla\cdot\vec{A} = -\frac{\rho}{\varepsilon}.$$

From Ampere's law comes

$$\nabla\times\nabla\times\vec{A} = \mu\vec{J} + j\omega\mu\varepsilon_c(-\nabla\Phi - j\omega\vec{A}).$$

After application of the vector identity

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

this becomes

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu\vec{J} - j\omega\mu\varepsilon_c\left(\nabla\Phi + j\omega\vec{A}\right)$$

or

$$\nabla^2\vec{A} + k^2\vec{A} - \nabla\left(\nabla\cdot\vec{A} + \frac{jk^2}{\omega}\Phi\right) = -\mu\vec{J}$$

where

$$k^2 = \omega^2\mu\varepsilon_c$$

defines the wavenumber in the unbounded medium. Now

$$\nabla\times\vec{A} = \vec{B}$$

and

$$\nabla\cdot\vec{A} = -\frac{jk^2}{\omega}\Phi$$

is a convenient choice known as the *Lorentz condition*. This leads to

$$\nabla^2\Phi + k^2\Phi = -\frac{\rho}{\varepsilon}$$

$$\nabla^2\vec{A} + k^2\vec{A} = -\mu\vec{J}$$

which are a pair of inhomogeneous *Helmholtz equations* for $(\Phi, \vec{A})$ maintained by $(\vec{\rho}, \vec{J})$. These equations must be solved for $\Phi, \vec{A}$ in order to determine $\vec{E}, \vec{B}$. It should be noted that $\vec{E}$ is maintained by both ρ and $\vec{J}$, while $\vec{B}$ has only $\vec{J}$ as a source. The inhomogeneous Helmholtz equations above are second order partial differential equations, and are solved using the concept of Green's functions.

- **Green's function solution to Helmholtz equation**

The Helmholtz equations may be decomposed into scalar components

$$\nabla^2 A_\alpha + k^2 A_\alpha = -\mu J_\alpha$$

where α represents various component directions such as (x, y, z) in Cartesian coordinates. Now consider a generic form of the Helmholtz equation to be solved

$$\nabla^2\psi(\vec{r}) + k^2\psi(\vec{r}) = -s(\vec{r})$$

where $\psi(\vec{r})$ is a wave function representative of $\Phi(\vec{r})$ or $A_\alpha(\vec{r})$, and $s(\vec{r})$ is a source density representative of $\rho(\vec{r})$ or $J_\alpha(\vec{r})$. This general Helmholtz equation can be solved using the *Green's function method*. The Green's function is a solution to the partial differential equation when the source term is a point source. This method involves a two-step process. First, the solution for a unit point source excitation located at an arbitrary point $\vec{r}^{\,/}$ is determined. Then, a general solution is constructed by the linear superposition of many weighted point-source responses.

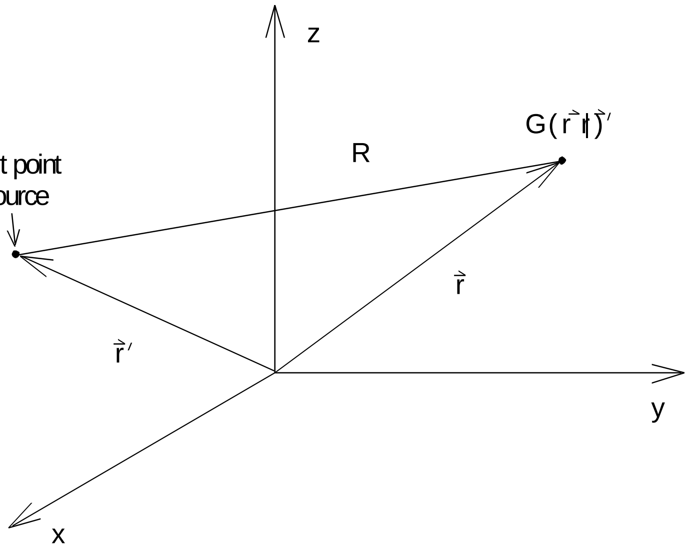

Figure 17.Unit point source excitation.

For the case of point source excitation, let

$$s(\vec{r}) \rightarrow \delta(\vec{r} - \vec{r}^{\,\prime})$$

and

$$\Phi(\vec{r}) \rightarrow G(\vec{r}|\vec{r}^{\,\prime})$$

which leads to

$$\nabla^2 G(\vec{r}|\vec{r}^{\,\prime}) + k^2 G(\vec{r}|\vec{r}^{\,\prime}) = -\delta(\vec{r} - \vec{r}^{\,\prime}) .$$

Here $G(\vec{r}|\vec{r}^{\,\prime})$ is the Green's function, which represents the response at $\vec{r}$ due to a unit point source located at $\vec{r}^{\,\prime}$, and $\delta(\vec{r})$ is a three-dimensional Dirac delta function. The Dirac delta has the following properties:

i.

$$\delta(\vec{r}) = \begin{cases} 0 & \ldots for\ \ \vec{r} \neq 0 \\ \infty & \ldots for\ \ \vec{r} = 0 \end{cases}$$

where $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$

ii.

$$\int_V \delta(\vec{r})dv = \begin{cases} 1 \ \ldots \ if \ V \ includes \ \vec{r} = 0 \\ 0 \ \ldots \ otherwise \end{cases}$$

iii.

$$\int_V f(\vec{r})\,\delta(\vec{r})\,dv = \begin{cases} f(0) \ \ldots \ if \ 0 \in V \\ 0 \ \ldots \ otherwise \end{cases}$$

Now if a tiny element of the source at location $\vec{r}^{\,/}$ is described by

$$s(\vec{r}^{\,/})\,dv^{/}$$

then

$$d\psi(\vec{r}) = G(\vec{r}\,|\,\vec{r}^{\,/})\,s(\vec{r}^{\,/})\,dv^{/}$$

by the definition of the Green's function, and the total wave function can be found by superposition

$$\psi(\vec{r}) = \int_V s(\vec{r}^{\,/})G(\vec{r}|\vec{r}^{\,/})dv^{/}\,.$$

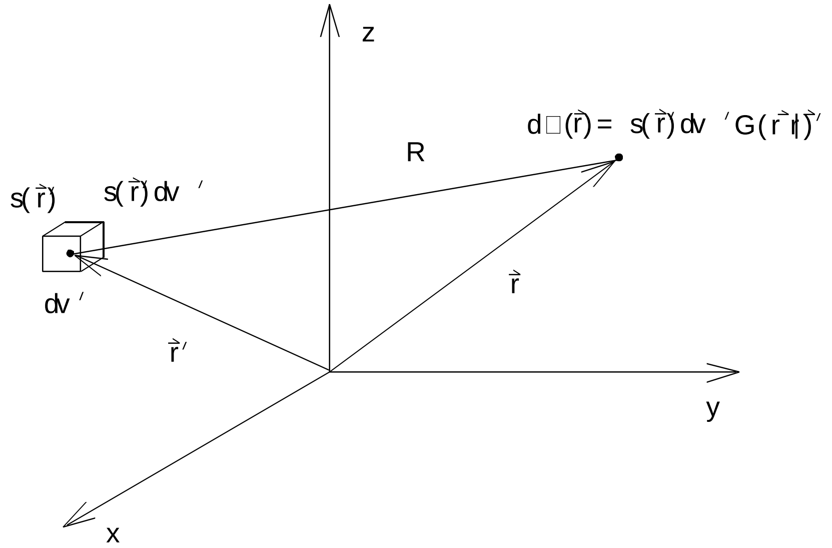

Figure 18.Distributed source excitation.

- <u>check</u>: Does this solution satisfy the Helmholtz equation? Substitution of the Green's function solution into the Helmholtz equation gives

$$(\nabla^2 + k^2)\psi(\vec{r}) = (\nabla^2 + k^2)\int_V s(\vec{r}^{\,/})\, G(\vec{r}|\vec{r}^{\,/})\, dv^{/}$$

$$= \int_V s(\vec{r}^{\,/})(\nabla^2 + k^2)\, G(\vec{r}|\vec{r}^{\,/})\, dv^{/}$$

$$= -\int_V s(\vec{r}^{\,/})\,\delta(\vec{r} - \vec{r}^{\,/})\, dv^{/}$$

$$= - s(\vec{r}^{\,/}) \,.$$

Thus it is seen that ψ satisfies the Helmholtz equation. Now the Green's function must be determined. The Green's function satisfies

$$(\nabla^2 + k^2)\, G(\vec{r}|\vec{r}^{\,/}) = -\delta(\vec{r} - \vec{r}^{\,/})$$

and

$$G(\vec{r}|\vec{r}^{\,/}) = G(\vec{r} - \vec{r}^{\,/}) = G(R)$$

because G depends only on the distance $R = |\vec{r}^{\,/} - \vec{r}^{\,/}|$ from the source points to the field point. This gives

$$(\nabla^2 + k^2)\, G(R) = -\delta(R) \,.$$

In spherical coordinates, this becomes

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial G}{\partial R}\right) + k^2 G = 0$$

which is equivalent to

$$\frac{1}{R}\frac{\partial^2}{\partial R^2}(RG) + k^2 G = 0 \qquad \ldots for\ \ R \neq 0$$

or

$$\frac{\partial^2}{\partial R^2}(RG) + k^2 (RG) = 0 \qquad \ldots for\ \ R \neq 0\,.$$

This expression has solution

$$RG = Ae^{-jkR} + Be^{+jkR}$$

therefore

$$G(R) = A\frac{e^{-jkR}}{R} + B\frac{e^{+jkR}}{R} \qquad \ldots for\ \ R \neq 0\,.$$

G(R) physically represents waves traveling outward from the source (no sources exist at infinity due to the Sommerfeld radiation condition), consequently, B=0 and

$$G(R) = A\frac{e^{-jk|\vec{r}-\vec{r}^{\,\prime}|}}{|\vec{r}-\vec{r}^{\,\prime}|}\,.$$

Now the constant 'A' must be determined. 'A' can be chosen to give the correct behavior of G(R) at R=0. This behavior is determined by examining a small spherical surface of radius ε surrounding the point R=0, and allowing ε to approach zero ($\varepsilon \to 0$). The defining equation for G is integrated over the small spherical volume region V_ε which is centered at the location of the δ-function unit point-source singularity

$$\int_{V_\varepsilon} \nabla\cdot\nabla G(r)\,dv^{\prime} + k^2 \int_{V_\varepsilon} G(r)\,dv^{\prime} = -\int_{V_\varepsilon} \delta(r)\,dv^{\prime}\,.$$

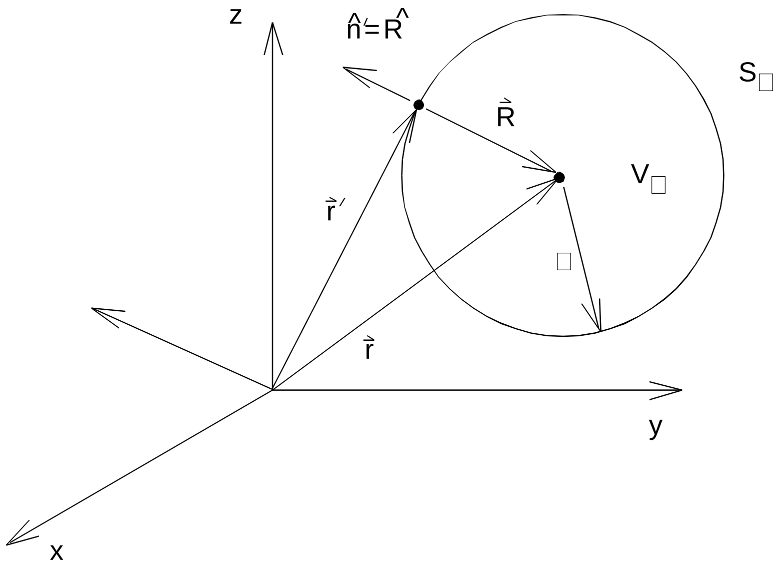

Figure 19.Spherical volume region about source point singularity.

Application of the divergence theorem yields

$$\oint_{S_\varepsilon} \hat{n}\cdot\nabla G(R)\,ds' + k^2\int_0^{\varepsilon} A\frac{e^{-jkR}}{R}4\pi R^2 dR = -1 .$$

Examining the behavior of the integrals as the radius of the sphere approaches zero leads to

$$\lim_{\varepsilon\to 0} k^2\int_0^{\varepsilon} A\frac{e^{-jkR}}{R}\,4\pi R^2 dR = k^2 A\lim_{\varepsilon\to 0}\int_0^{\varepsilon} 4\pi R\,dR = 0$$

$$\hat{n}\cdot\nabla G(R) = \hat{R}\cdot G(R) = \frac{\partial G(R)}{R} = \frac{\partial}{\partial R}\left(A\frac{e^{-jkR}}{R}\right)$$

$$= -A\left(\frac{1+jkR}{R^2}\right)e^{-jkR}$$

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon}\hat{n}\cdot\nabla G(R)\,ds' = -A\lim_{\varepsilon\to 0}\int_{S_\varepsilon}\left(\frac{1+jkR}{\varepsilon^2}\right)e^{jk\varepsilon}ds'$$

$$= -A \lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{ds'}{\varepsilon^2} = -\lim_{\varepsilon \to 0} \frac{A}{\varepsilon^2} 4\pi\varepsilon^2$$

$$= -4\pi A .$$

Therefore

$$-4\pi A + 0 = -1$$

or

$$A = \frac{1}{4\pi}$$

and

$$G(R) = \frac{e^{-jkR}}{4\pi R} .$$

Thus the Green's function is found to be

$$G(\vec{r}|\vec{r}^{\,\prime}) = G(\vec{r} - \vec{r}^{\,\prime}) = \frac{e^{-jk|\vec{r} - \vec{r}^{\,\prime}|}}{4\pi|\vec{r} - \vec{r}^{\,\prime}|} = \frac{e^{-jkR}}{4\pi R}$$

and the associated wave function is then

$$\psi(\vec{r}) = \int_V s(\vec{r}^{\,\prime})\frac{e^{-jkR}}{4\pi R}dv^{\prime}.$$

The substitution of appropriate potential and source terms then gives solutions for the potential functions

$$\Phi(\vec{r}) = \frac{1}{4\pi\varepsilon}\int_V \rho(r^{\prime})\frac{e^{-jkR}}{R}dv^{\prime}$$

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi}\int_V \vec{J}(r^{\prime})\frac{e^{-jkR}}{R}dv^{\prime}.$$

- **Retarded EM potential functions**

Electromagnetic disturbances do not travel instantaneously through space. A certain amount of time is required for EM waves to propagate and for effects due to time-changing charge and current distributions to be detected. The values of the vector and scalar potentials at some distance R from a source and at some time t depend of the condition of the source at an earlier time (t - R/v), where v is the velocity of wave propagation. The effects due to this finite propagation time are typically ignored for dc and quasi-static cases. However, these effects must be taken into account when the dimensions of a circuit or device are on the order of a wavelength long. Such phenomena are described by the so-called *retarded potential functions*.

Consider a localized time-harmonic current density

$$\vec{J}(\vec{r},t) = Re\left\{\vec{J}(\vec{r})e^{j\omega t}\right\}$$

where $Re\{\ \}$ denotes the real part of the complex quantity.

The associated vector potential is given by

$$\vec{A}(\vec{r},t) \;=\; Re\left\{\vec{A}(\vec{r})e^{j\omega t}\right\} \;=\; Re\left\{e^{j\omega t}\frac{\mu}{4\pi}\int_V \vec{J}(\vec{r}^{\,\prime})\frac{e^{-jkR}}{R}dv^{\prime}\right\}$$

$$=\; Re\left\{\frac{\mu}{4\pi}\int_V \frac{\vec{J}(\vec{r}^{\,\prime})\,e^{j\omega(t\,-\,kR/\omega)}}{R}dv^{\prime}\right\}$$

$$=\; \frac{\mu}{4\pi}\int_V \frac{\vec{J}(\vec{r}^{\,\prime},\, t^{\prime} = t - R/v)}{R}dv^{\prime}\,.$$

The potential $\vec{A}(\vec{r},t)$ at location $\vec{r}$ and time t is the superposition of contributions by sources $\vec{J}(\vec{r}^{\,\prime}, t - R/v)\,dv^{\prime}$ at location $\vec{r}^{\,\prime}$ which occurred at the earlier time $t - R/v$. The retardation time $t = R/v$ is the time required for the wave disturbance to propagate through distance R from the source point $\vec{r}^{\,\prime}$ to field point $\vec{r}$ with propagation velocity $v = \omega/k$. The fields at points removed from the source are thus described by

$$\Phi(\vec{r},t) \;=\; \frac{1}{4\pi\varepsilon}\int_V \frac{\rho(\vec{r}^{\,\prime},\, t - R/v)}{R}dv^{\prime}$$

which is the *retarded EM scalar potential*, and

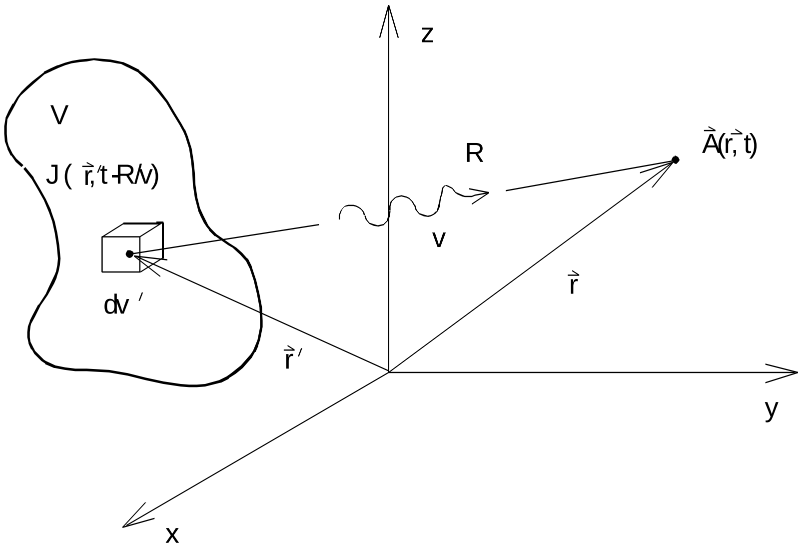

Figure 20.Time-retarded potential function.

$$\vec{A}(\vec{r},t) = \frac{\mu}{4\pi}\int_V \frac{\vec{J}(\vec{r}^{\,/},\, t - R/v)}{R}\,dv^{/}$$

which is the *retarded EM vector potential*.

- **Spherical waves**

Now consider the element of vector potential contributed by $\vec{J}(\vec{r}^{\,/})$ in $dv^{/}$ at location $\vec{r}^{\,/}$

$$d\vec{A}(\vec{r}\,) = \frac{\mu}{4\pi}\vec{J}(\vec{r}^{\,/})\,dv^{/}\,\frac{e^{-jkR}}{R}\,.$$

The instantaneous value of this element is given by

$$d\vec{A}(\vec{r},t) = d\vec{A}(\vec{r}\,)\,e^{j\omega t} = \frac{\mu\,\vec{J}(\vec{r}^{\,/})\,dv^{/}}{4\pi}\,\frac{e^{j\omega(t - kR/\omega)}}{R}$$

$$= \frac{\mu\,\vec{J}(\vec{r}^{\,/})\,dv^{/}}{4\pi}\,\frac{e^{j\omega(t - R/v)}}{R}\,.$$

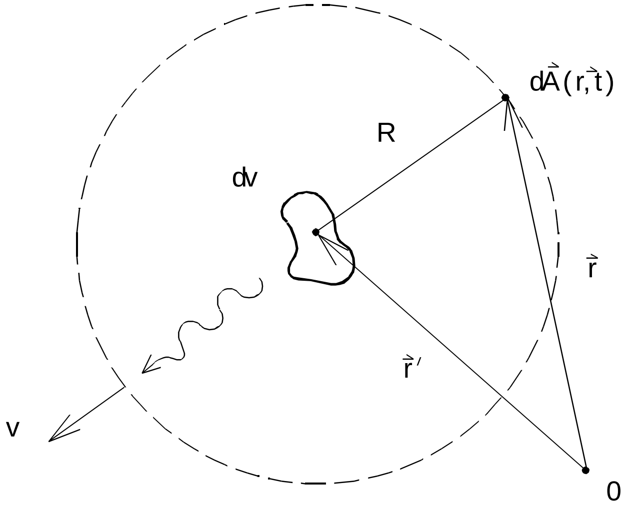

Figure 21.Spherical EM wave.

This expression describes a wave whose points of constant amplitude and phase lie at a location described by $R = |\vec{r} - \vec{r}^{\,\prime}| =$ constant. This is therefore a wave with spherical wavefronts. Points of constant phase are described by

$$e^{j\omega(t - R/v)} = constant$$

which implies

$$t - R/v = constant\,.$$

The time derivative of a constant is zero, therefore

$$1 - \frac{1}{v}\frac{dR}{dt} = 0$$

or

$$\frac{dR}{dt} = v_p = \frac{\omega}{k}$$

which describes the phase velocity of the spherical waves. Now let R_1 and R_2 be equiphase points, then

$$e^{-jkR_2} = e^{-jkR_1}$$

or

$$e^{-jk(R_2 - R_1)} = 1\,.$$

This requires

$$k(R_2 - R_1) = 2n\pi$$

or

$$(R_2 - R_1) = n\frac{2\pi}{k} = n\lambda$$

where

$$\lambda = \frac{2\pi}{k}$$

is the distance between adjacent spherical wavefronts (wavelength).

References

1. Cheng, D., *Field and Wave Electromagnetics*, Addison-Wesley Publishing Company, second edition, 1989.

2. Ramo, Whinnery, and Van Duzer, *Fields and Waves in Communication Electronics*, John Wiley & Sons, second edition, 1984.

3. Paul, C., *Introduction to Electromagnetic Compatibility*, John Wiley & Sons, 1992.

4. Irwin, J. D., *Basic Engineering Circuit Analysis*, Macmillan Publishing Company, third edition, 1989.

5. D. Nyquist and E. Rothwell, class notes for EE 305, EE 306, and EE 435, Michigan State University, Dept. of Electrical Engineering.

Appendix A: Potentially Useful Stuff

- **Divergence theorem**

$$\int_V \nabla \cdot \vec{A}\, dv = \oint_S \vec{A} \cdot d\vec{s}$$

- **Stoke's theorem**

$$\int_S (\nabla \times \vec{A}) \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

- **Other useful vector identities**

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\nabla(\psi V) = \psi \nabla V + V \nabla \psi$$

$$\nabla \cdot (\psi \vec{A}) = \psi \nabla \cdot \vec{A} = \vec{A} \cdot \nabla \psi$$

$$\nabla \times (\psi \vec{A}) = \psi \nabla \times \vec{A} = \nabla \psi \times \vec{A}$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\nabla \cdot \nabla V = \nabla^2 V$$

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla \times \nabla V = 0$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

- **Gradient, divergence, curl, and Laplacian operators**

  - Cartesian coordinates (x, y, z)

$$\nabla V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

  - Cylindrical coordinates (r, φ, z)

$$\nabla V = \hat{r}\frac{\partial V}{\partial r} + \hat{\varphi}\frac{\partial V}{r\partial \varphi} + \hat{z}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\varphi}{r\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & \hat{\varphi}r & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix} = \hat{r}\left(\frac{\partial A_z}{r\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \hat{\varphi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\varphi) - \frac{\partial A_r}{\partial \varphi}\right)$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}$$

- Spherical coordinates (R, θ, φ)

$$\nabla V = \hat{R}\frac{\partial V}{\partial R} + \hat{\theta}\frac{\partial V}{R\partial \theta} + \hat{\varphi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \varphi}$$

$$\nabla\cdot\vec{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

$$\nabla\times\vec{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{R} & \hat{\theta}R & \hat{\varphi}R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \varphi} \\ A_R & RA_\theta & (R\sin\theta)A_\varphi \end{vmatrix}$$

$$= \hat{R}\,\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial \varphi}\right] + \hat{\theta}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \varphi} - \frac{\partial}{\partial R}(RA_\varphi)\right] + \hat{\varphi}\,\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2}$$