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DESIGN OF PENDULUM ABSORBERS FOR TRANSVERSE VIBRATION ATTENUATION OF ROTATING BEAMS

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ABSTRACT

This paper considers the placement, sizing and tuning of centrifugal pendulum vibration absorbers for the reduction of transverse vibrations in rotating beams. A simplified model describing the linearized dynamics of a rotating beam with external excitation and attached absorbers is used for the analysis. A design strategy is offered wherein individual absorbers are designed to reduce vibration amplitudes and stress levels caused by troublesome resonances. It is shown that this procedure offers significant reduction in vibratory stresses, even in the case of excitations composed of multiple harmonics.

keywords: pendulum absorbers, transverse beam vibrations, resonances.

1 INTRODUCTION

In the dynamics of rotating and reciprocating machinery disturbance loads are often encountered which have frequencies that are proportional to the nominal rotational rate of the system. Such loads arise from gas pressure and inertial effects in internal combustion engines and from fluid loading in helicopter rotors and turbines. There is typically a base order of the excitation, denoted here as n, such that the fundamental frequency of excitation is $n\ \Omega$, where Ω denotes the mean rotational rate of the system. Such disturbances are a significant source of vibrations and are not conveniently handled by conventional tuned vibration absorbers, since they are tuned to a fixed frequency rather than to a fixed order. A successful means of reducing vibration levels in such cases is the *centrifugal pendulum vibration absorber*, or CPVA, which is a device that can be tuned to a given *order*, as opposed to a given frequency. Such devices have had great success in reducing vibration levels in helicopter rotors and light aircraft piston engines. In this paper, we demonstrate the utility of these devices for reducing transverse vibrations in rotating beams, such as blades in helicopter rotors and gas turbine engines.

General background information on the CPVA can be found in the treatise of Ker Wilson (1968) or the recent survey of Shaw and Lee (1995). Examples of specific application studies can be found in the following works: Borowski *et al.* (1991), Cronin (1992), Denman (1992) and Lee and Shaw (1995) for automotive applications, Wachs (1973) and Miao and Mouzakis (1980) for helicopter rotors, and Murthy and Hammond (1981) and Hamouda and Pierce (1984) for rotor blades. The latter two studies are most relevant to the present work, in that pendulum absorbers are considered for vibration suppression in rotating blades. These latter two works considered specific models for CPVA's applied to helicopter blades and carried out detailed numerical parameter studies, demonstrating the effectiveness of absorbers in reducing vibration and stress levels. The present work focuses on a more basic system, considers more details of the beam frequency response, and examines the selection of pendulum mass ratio and absorber placement and their effects on the stresses induced in the rotating beam. Also, a systematic design strategy based on the equations of motion is offered.

We consider a cantilever beam attached to a rigid hub that is rotating at a fixed speed Ω about the hub's geometric center. (This beam may be one of an assembly.) The beam is subjected to a distributed load that is periodic in time

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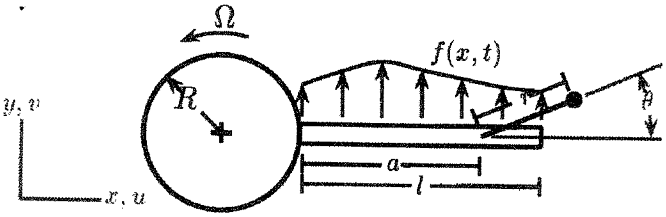

Figure 1. A schematic diagram of the basic system with a single absorber.

with period $2\pi/\Omega$. This load may be composed of several temporal harmonics of significant amplitude. Our goal is to show that with proper tuning and location one can significantly reduce the steady-state vibratory stress levels in the beam at selected orders. The idealized model considered herein demonstrates the main principles and can be used as a first step in the design of actual devices.

It should be noted that the analysis presented here is based on the assumptions of linearized motion and zero damping. These results should be adequate for initial design purposes, as it is highly desirable (and generally feasible) to keep the dissipation of CPVA devices quite low and maintain their frequencies in the linear range. This can be accomplished even for large-amplitude motions of the absorber by utilizing a bifilar configuration that employs a noncircular paths for the CG of the absorber (Madden, 1980; Denman, 1992; Lee and Shaw, 1995).

2 THE BASIC SYSTEM

The physical system under consideration is depicted in Fig. 1. It consists of a rotating rigid hub of radius R with an attached flexible beam of length l, to which is attached at a point of distance $a + R$ from the center of rotation a pendulum absorber with effective length r. The parameters ρ, m, E, I, and A denote the density, mass per unit length, Young's modulus, second moment of area and cross-sectional area of the cantilever beam, respectively. M is the mass of pendulum absorber, which is idealized as a point mass. The base is taken to rotate with fixed speed Ω and the beam is subjected to the distributed load $f(x,t)$ that rotates with the beam. The coordinate system $x - y$ and the deflections $u - v$ are fixed in the frame of reference that rotates with the system and are aligned with the unstressed beam configuration, as shown in Fig. 1.

Using Hamilton's principle, and considering only the motion in the $x - y$ plane, the linearized p.d.e's of motion for the transverse and longitudinal beam vibrations and the motion of the absorber pendulum can be derived. By ignoring the longitudinal inertia of the beam and assuming that displacements and velocities scale as follows: $R \gg u$ and $\Omega(R + x) \gg v_{,t}$, the longitudinal dynamics can be incorporated into the transverse dynamics by some proper integrations and utilization of the corresponding boundary conditions. Using this approach, for the case where E and I are constant, the non-dimensional equations of motion of the system can be expressed as follows:

$$\tilde{v}_{,\xi\xi\xi\xi} + \frac{\rho I}{ml^2}\tilde{\Omega}^2\tilde{v}_{,\xi\xi} - [F(\xi)\tilde{v}_{,\xi}]_{,\xi} - \tilde{\Omega}^2\tilde{v} + \tilde{v}_{,\tau\tau}$$
$$+\nu\delta(\xi - \frac{a}{l})(\tilde{v}_{,\tau\tau} - \tilde{\Omega}^2\tilde{v} - \frac{r}{l}\tilde{\Omega}^2\tilde{\Theta} + \frac{r}{l}\ddot{\tilde{\Theta}}) = \frac{l^3}{EI}f(\xi,\tau), \quad (1a)$$
$$\ddot{\tilde{\Theta}} + \frac{R+a}{r}\tilde{\Omega}^2\tilde{\Theta} + \frac{l}{r}\tilde{v}_{,\tau\tau}\,|_{\xi=\frac{a}{l}} - \frac{l}{r}\Omega^2\tilde{v}\,|_{\xi=\frac{a}{l}} = 0, \quad (1b)$$

with boundary conditions,

$$\tilde{v}(0,\tau) = 0,\ \tilde{v}_{,\xi}(0,\tau) = 0,\ \tilde{v}_{,\xi\xi}(1,\tau) = 0, \quad (2a)$$
$$\tilde{v}_{,\xi\xi\xi}\,|_{\xi=1} + \frac{\rho I}{ml^2}\tilde{\Omega}^2\tilde{v}_{,\xi}\,|_{\xi=1} = 0, \quad (2b)$$

wherein the following dimensionless parameters and functions have been defined:

$$F(\xi) = \frac{1}{2}\tilde{\Omega}^2\left[(\frac{R}{l}+1)^2 - (\frac{R}{l}+\frac{\xi}{l})^2\right] +$$
$$\nu\tilde{\Omega}^2\frac{R+r+a}{l}\left[1 - \mu(\xi - \frac{a}{l})\right],$$
$$\xi = \frac{x}{l},\ \tilde{v} = \frac{v}{l},\ \tau = t\omega,\ \omega^2 = \frac{EI}{ml^4},\ \tilde{\Omega} = \frac{\Omega}{\omega},\ \nu = \frac{M}{ml}.$$

Note that ξ represents the dimensionless coordinate along the beam,. $\tilde{v}$ represents the dimensionless transverse displacement of the beam, τ is time normalized by the beam characteristic frequency ω, $\tilde{\Omega}$ is the similarly normalized rotation rate, and ν is the ratio of pendulum mass to beam mass (typically much less than one in value).

These equations will be analyzed in terms of the dimensionless parameters in order to determine the performance capabilities of the absorber in reducing transverse beam vibrations.

3 THE ROTATING BEAM WITHOUT ABSORBERS

3.1 Free Vibrations

In order to properly assess the effectiveness of the pendulum absorber one needs to compare the vibration amplitudes of the system with and without the absorber under given forcing conditions. To this end we investigate here

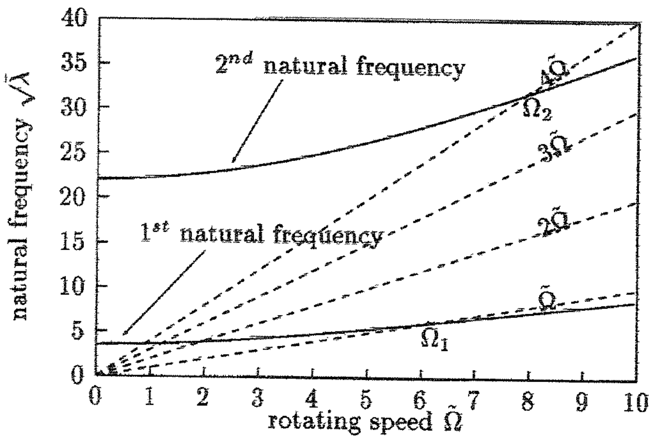

Figure 2. Natural frequencies and excitation frequencies as a function of the rotation speed (a Campbell diagram); $\frac{R}{l} = \frac{1}{3}$, $\frac{EI}{ml^4} = 3.537 \times 10^3$, $\frac{\rho I}{ml^2} = 7.217 \times 10^{-3}$.

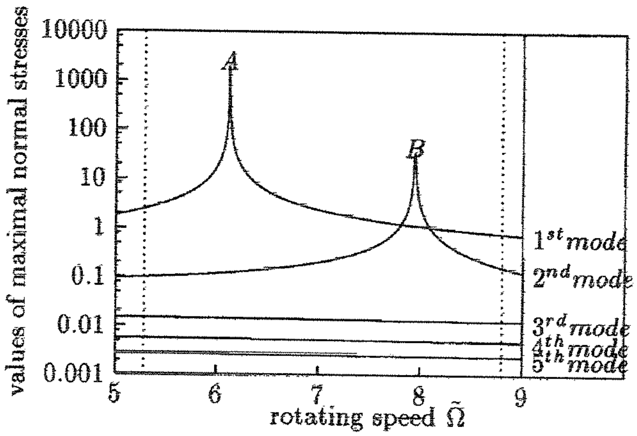

Figure 3. Maximum normal stress levels for the first five modes versus rotation speed for the beam with no absorbers; $f(\xi,\tau) = sin(\tilde{\Omega}_1\tau) + sin(4\tilde{\Omega}_2\tau)$; $\tilde{\Omega}_1 = 6.121$, $\tilde{\Omega}_2 = 7.950$, $\frac{R}{l} = \frac{1}{3}$, $\frac{EI}{ml^4} = 3.537 \times 10^3$, $\frac{\rho I}{ml^2} = 7.217 \times 10^{-3}$.

the free and forced vibrations of the rotating beam without an absorber. The free vibrations are considered first, as the corresponding mode shapes are used in the application of Galerkin's method (Bisplinghoff *et al*, 1957) for the subsequent analysis of the forced system, with and without absorbers.

For the unforced case, the natural frequencies and normal modes of the system can be obtained by standard methods (Meirovitch, 1967). The numerical solution of this eigenvalue/eigenfunction problem yields a set of natural frequencies $\sqrt{\lambda_i}$ and mode shapes that depend on the rotational speed $\tilde{\Omega}$. These frequencies increase as a function of $\tilde{\Omega}$ due to centrifugal stiffening effects. Figure 2 shows, for a rectangular beam with typical dimensions taken from a turbine blade, the first two natural frequencies, Ω_1 and Ω_2, as functions of $\tilde{\Omega}$.

3.2 Forced Vibrations

Vibration problems associated with resonances in systems such as turbine blades occur when the excitation has a frequency component that is close to one of the $\tilde{\Omega}$-dependent natural frequencies. In these rotating systems the excitation contains harmonic components with frequencies that are multiples of the rotation rate, and a component of the excitation with frequency $n\,\tilde{\Omega}$ is said to be of order n. Figure 2 shows as dashed lines the first four harmonic orders for a typical excitation. It is clear that for the example being considered, there are two trouble spots in the speed range corresponding to $\tilde{\Omega} = 3 - 10$. Near $\tilde{\Omega} = 6$ the first mode is excited by the first order component of the excitation ($n = 1$) while near $\tilde{\Omega} = 8$ the second mode of the beam is excited by the fourth order excitation component ($n = 4$). (Note that the spin-up problem is also evident here, as the lower beam modes must necessarily pass through high orders of the excitation. This, fortunately, is typically a transient operating regime.)

For purposes of demonstration, consider a simple case where the excitation is uniform along the beam and composed of two harmonics, as follows,

$$f(\xi,\tau) = a_1 \sin(\tilde{\Omega}\tau) + a_4 \sin(4\,\tilde{\Omega}\tau). \tag{3}$$

The steady-state frequency response of the first five beam modes to this excitation with $a_1 = a_4 = 1$ is given in figure 3. ("Modes" here refer to the natural modes of vibration of the rotating beam.) Here only the operating range of interest is shown, $\tilde{\Omega}$ from 5.0 to 9.0, corresponding approximately to a range 3000—5000rpm. The measure of vibration amplitude used is the maximum nondimensional bending stress along the beam (typically at the root) for each modal component. Here the two resonances are evident — point A for the first mode and point B for the second mode.

This response will be used as a basis for comparison.

4 APPLICATION OF A SINGLE ABSORBER

When adding a vibration absorber to a flexible structure there are many design parameters to be selected. Of particular importance are the tuning of the absorber, its location along the beam, and its mass. In this section we consider the application of a single absorber tuned to remove the first mode resonance, that is, at point A in figure 3.

4.1 Steady-State Response

Our goal here is to determine the steady-state system response and use these results to design the absorber. In order to proceed we first return to the equations of motion for the beam/absorber system and use separation of variables. Since we are only concerned with resonance point A we will determine the response to the general first-order excitation given by

$$f(\xi,\tau) = A(\xi)\ g(\tau) = A(\xi)\ \sin(\tilde{\Omega}\tau). \tag{4}$$

For the spatial nature of the beam steady-state response, the expansion theorem is employed wherein the modes of the rotating beam are used as basis functions (Meirovitch, 1967). Therefore, we express the beam and pendulum dynamics in the following form,

$$\tilde{v}(\xi,\tau) = V(\xi)\ g(\tau),$$

$$V(\xi) = \sum_{i=1}^{\infty} \alpha_i \varphi_i(\xi), \tag{5}$$

$$\tilde{\Theta}(\tau) = \tilde{\theta} g(\tau), \tag{6}$$

where $V(\xi)$ is the steady-state beam configuration (including both amplitude and shape information), $\varphi_i(\xi)$ are the eigenfunctions of the rotating beam, α_i are the coefficients that describe the shape in terms of the $\varphi_i(\xi)$, and $\tilde{\theta}$ is the steady-state amplitude of pendulum motion. Note that $\tilde{\theta}$ must be kept small if the linearized theory used here is to be valid. This can be accomplished by using sufficient absorber mass, although this approach has practical limitations. (A more interesting and practical solution is to use tautochronic absorbers; see Denman, 1992 or Lee and Shaw, 1995). Also, note that this choice of basis functions is not optimal, as the eventual solution will have kinks at the point of pendulum attachment that could be better handled by other functions. However, this solution converges quite quickly and is satisfactory for present purposes.

The above expressions are substituted into the equations of motion (1) and the time dependence is removed by eliminating $g(\tau)$. The resulting equations, which are to be solved for θ and the α_i, are given by

$$\sum_{i=1}^{\infty} \alpha_i \varphi_i'''' + \frac{\rho I}{ml^2}\tilde{\Omega}^2 \sum_{i=1}^{\infty} \alpha_i \varphi_i'' - \left[F(\xi)\sum_{i=1}^{\infty} \alpha_i \varphi_i'\right]' -$$

$$\tilde{\Omega}^2 \sum_{i=1}^{\infty} \alpha_i \varphi_i - n^2\tilde{\Omega}^2 \sum_{i=1}^{\infty} \alpha_i \varphi_i$$

$$-\nu\tilde{\Omega}^2(n^2+1)\delta(\xi - \frac{a}{l})(\sum_{i=1}^{\infty} \alpha_i + \frac{r}{l}\tilde{\theta}) = \frac{l^3}{EI}A(\xi), \tag{7a}$$

$$(\frac{R+a}{r} - n^2)\tilde{\theta} - \frac{l}{r}(n^2+1)\sum_{i=1}^{\infty} \alpha_i \varphi_i(a) = 0. \tag{7b}$$

These equations can be projected onto the eigenfunctions and simplified for solution. However, we proceed a different line that offers the desired outcome. Specifically, a zero steady-state vibrational amplitude is imposed at the position at which the pendulum is suspended. That is, $V(a) = 0$, or, equivalently, $\sum_{i=1}^{\infty} \alpha_i \varphi_i(a) = 0$. Using this condition in equation (7*b*) immediately yields a tuning condition which sets the desired pendulum length as follows,

$$r = \frac{R+a}{n^2}. \tag{8}$$

This condition sets the natural frequency of the pendulum, if attached to a rigid beam, to be equal to $n\Omega$, thereby making it equal to the frequency of the applied load, just as required for a tuned absorber. Note that for low order harmonics the effective length of the pendulum can be quite large. However, there exist kinematic designs that offer large effective pendulum lengths in small spaces; the treatise by KerWilson (1964) offers a thorough treatment and examples of these devices.

In order to determine the steady-state beam shape, equation (7*a*) is projected onto each eigenfunction $\varphi_i(\xi)$, yielding

$$(\lambda_i - n^2\tilde{\Omega}^2)\alpha_i - \frac{r}{l}\nu\tilde{\Omega}^2\varphi_i(a)(n^2+1)\tilde{\theta} = \tilde{A}_i, \tag{9}$$

where $\tilde{A}_i = \int_0^1 \frac{l^3}{EI}A(\xi)\varphi_i(\xi)d\xi$. This equation and the condition $\sum_{i=1}^{\infty} \alpha_i \varphi_i(a) = 0$ are used to solve for the α_i and $\tilde{\theta}$. This can be accomplished by solving equation (9) for α_i in terms of $\tilde{\theta}$, substituting the result into $\sum_{i=1}^{\infty} \alpha_i \varphi_i(a) = 0$ and solving for $\tilde{\theta}$. This is straightforward and the details are not provided here.

4.2 Selection of Absorber Parameters

There are three parameters which must to be decided for the absorber: the length of the pendulum r, the location of the absorber a, and the mass ratio ν. In this section we carry out calculations for a uniform unit loading on the

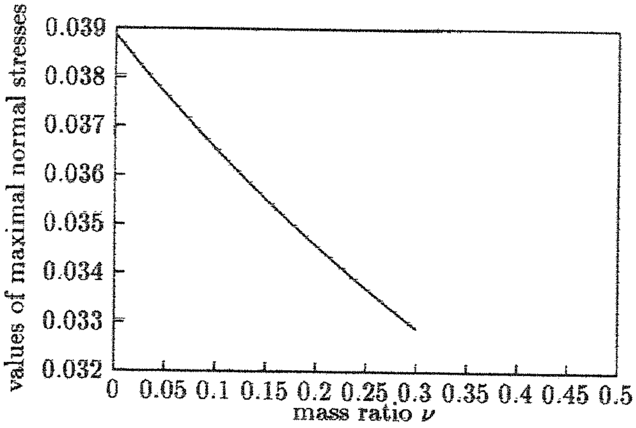

Figure 4. Maximum normal stress level versus mass ratio; $\tilde{\Omega} = 5.50$, $\frac{R}{l} = \frac{1}{3}$, $\frac{a}{l} = 0.73$, $\frac{EI}{ml^4} = 3.537 \times 10^3$, $\frac{\rho I}{ml^2} = 7.217 \times 10^{-3}$.

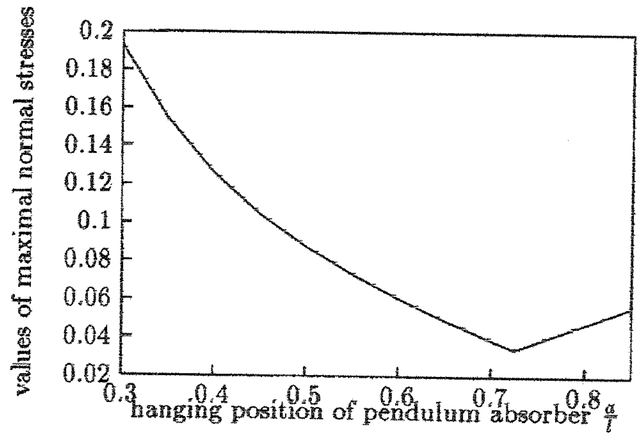

Figure 5. Maximum normal stress level versus absorber location; $f(\xi, \tau) = sin(\tilde{\Omega}_1 \tau)$; $\tilde{\Omega}_1 = 6.121$, $\frac{R}{l} = \frac{1}{3}$, $\nu = 0.1$, $\frac{EI}{ml^4} = 3.537 \times 10^3$, $\frac{\rho I}{ml^2} = 7.217 \times 10^{-3}$.

beam, that is, $A(\xi) = 1$. The dependence of the results on this special case are discussed subsequently.

It is generally known that the effectiveness of the absorber is monotonically increased in terms of transverse vibration reduction as the absorber mass is increased, independent of the form of the forcing spatial distribution. This is demonstrated in figure 4 where we have plotted the maximum bending stress along the beam as a function of the mass ratio ν, as obtained from the steady-state solution at the resonant forcing condition for the unit uniform loading example. (The value of a used for this example is determined below). However, there are practical limits to the amount of mass that can be used, as a large mass will increase the mass of the overall system and will increase longitudinal stresses. In this study we use $\nu = 0.1$, although actual limits on ν will be imposed by factors particular to the application at hand.

The position of the absorber a and the pendulum length r are constrained by the tuning condition (8), and thus only one is an independent design parameter. Here we vary a and use (8) to compute the corresponding value of r.

In order to select the absorber attachment position a we have computed the peak bending stress level along the beam as a function of $\frac{a}{l}$ for the resonant forcing condition using the steady-state solution given above with $A(\xi) = 1$. The results are given in figure 5, which shows that $\frac{a}{l} = 0.73$ offers the best placement of the absorber. The resulting stress distribution for this case is shown in figure 6. Note that for the optimal value of a, the stress at the attachment point is equal to that at the root. In fact, this is the source of the discontinuous nature of the peak stress shown in figure 5, since as $\frac{a}{l}$ passes through this value the location of the peak stress jumps from the root of the beam to the attachment point. (This quality of a "min-max" optimal solution is quite common. Also note that this optimization problem is very similar to one in which one adds a simple support at some point to be chosen along the length of a nonrotating cantilever beam in order to reduce vibration levels for a given forcing.)

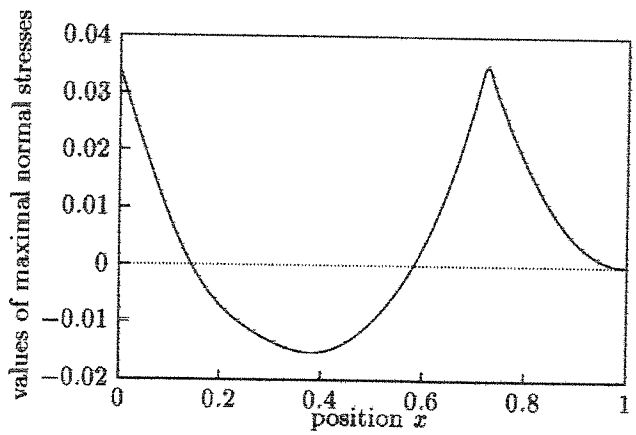

Figure 6. Stress distribution along the beam under the resonant excitation; $\frac{a}{l} = 0.73$, and the mass ratio $\nu = 0.1$, under the resonant load $f(\xi, \tau) = sin(\tilde{\Omega}_1 \tau)$; $\tilde{\Omega}_1 = 6.121$, $\frac{R}{l} = \frac{1}{3}$, $\frac{EI}{ml^4} = 3.537 \times 10^3$, $\frac{\rho I}{ml^2} = 7.217 \times 10^{-3}$.

It is important to note that this optimal location depends very strongly on the spatial distribution of the loading, that is, on $A(\xi)$. For example, for the case of turbine blades, a concentrated force at the beam tip might be a

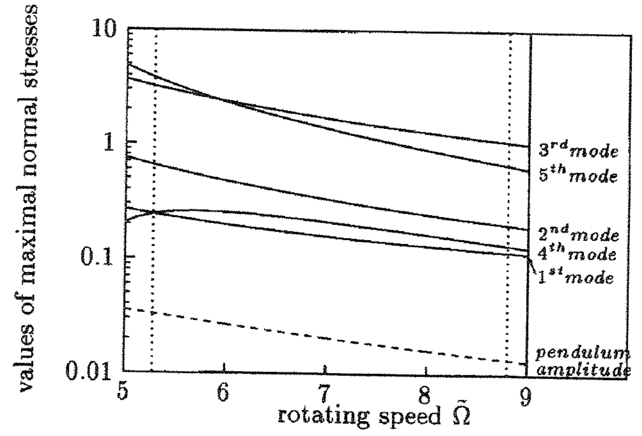

Figure 7. Maximum normal stress levels for the first five modes versus rotation speed for the beam with a single pendulum absorber; $f(\xi,\tau) = sin(\tilde{\Omega}_1\tau)$; $\tilde{\Omega}_1 = 6.121$; $\frac{a}{l} = 0.73$, $\nu = 0.1$, $\frac{R}{l} = \frac{1}{3}$, $\frac{EI}{ml^4} = 3.537\times10^3$, $\frac{\rho I}{ml^2} = 7.217\times10^{-3}$.

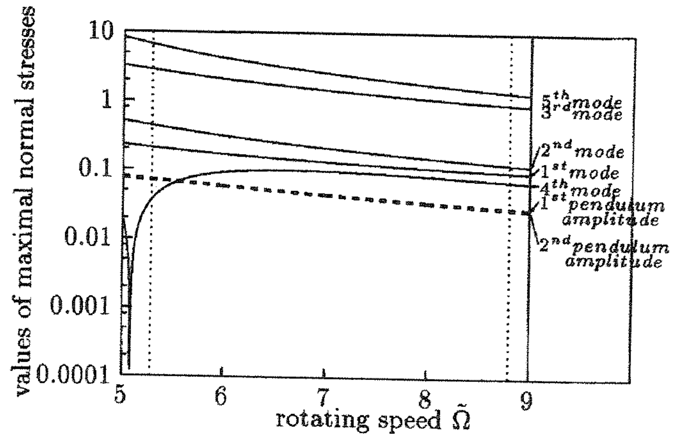

Figure 8. Maximum normal stress levels for the first five modes versus rotation speed for the beam with two pendulum absorbers; $\frac{a_1}{l} = 0.73$, $\frac{a_2}{l} = 0.390$, and the mass ratio $\nu_1 = 0.05$, $\nu_2 = 0.05$, under the resonant exciting load $f(\xi,\tau) = sin(\tilde{\Omega}_1\tau) + sin(4\tilde{\Omega}_2\tau)$; $\tilde{\Omega}_1 = 6.121$, $\tilde{\Omega}_2 = 7.950$, $\frac{R}{l} = \frac{1}{3}$, $\frac{EI}{ml^4} = 3.537\times10^3$, $\frac{\rho I}{ml^2} = 7.217\times10^{-3}$.

better first-order approximation and would yield different absorber placement.

A demonstration of the effectiveness of the absorber is given by the frequency response of the combined beam/absorber system, shown in figure 7. When compared with the system without the absorber, shown in figure 3, it is apparent that the absorber significantly reduces the overall levels of bending stress in the beam near and above resonance point A. (Recall that the resonance at point B is excluded in the present case since the excitation contains only a first order harmonic.) The resonance at point A has been completely eliminated, albeit at the expense of increased stress levels in the higher modes, particularly modes 3 and 5 in this case. The increased stress levels in higher modes are due to the fact that we have used the mode shapes of a rotating cantilever to express the steady-state vibration shape of the beam, whereas it now has a node at the absorber attachment point. It should be noted that vibrations in the higher modes will be more easily affected by structural and other sources of damping, resulting in a less severe increase than predicted in this undamped analysis.

It is also worth noting that the addition of such an absorber does not simply split the resonance peak into two peaks, as is the case with the usual tuned absorber, since in this rotational setting the absorber's tuning frequency automatically adjusts itself to the rate of rotation, and thus to the excitation frequency. Mathematically this is seen by noting that the tuning condition (8) does not depend on $\tilde{\Omega}$.

5 THE MULTI-ABSORBER CASE

The methodology outlined above can be applied to the case of multi-harmonic excitation by applying absorbers tuned to each bothersome harmonic. For example, in order to eliminate the resonance problems which occur for our example system for orders $n = 1$ and 4, two pendulum absorbers can be employed, one tuned to each order. This will eliminate the resonances that occur at points A and B in Figure 3.

In this case the forcing is taken to be $f(\xi,\tau) = \sin(\Omega\tau) + \sin(4\Omega\tau)$. The analysis proceeds exactly as in the single absorber case, although there are six parameters to be decided in this case. Here we proceed to select the parameters for each pendulum according to the scheme given for a single absorber, wherein each troublesome harmonic is treated individually. The overall design is achieved in this way. While this approach may not offer the optimal performance, it is simple and effective.

Thus, the tuning conditions are given by (8) with $n_1 = 1$ for the first absorber and $n_2 = 4$ for the second. We take the total absorber mass to be 10% of the beam, and it is split equally between the pendulums, i.e., $\nu_1 = \nu_2 = 0.05$ The desired location for the fourth order pendulum is $\frac{a_2}{l} = 0.39$, as this minimizes the bending stress in the beam when it is subjected to a uniformly distributed fourth order excitation. As before, we take $\frac{a_1}{l} = 0.73$ for the first-order absorber. With these parameters the frequency response is given in figure 8. When compared with the beam response without absorbers, figure 3, it is observed that both resonances are now eliminated. It is clear that this process can be extended to more absorbers and more resonances.

6 DISCUSSION

The present work offers a design strategy for the placement of absorbers, a detailed picture of how individual vibration modes are affected by the absorbers, and an analysis for multi-harmonic excitation over a range of operating speeds. It is seen that absorbers can be very effective in suppressing the resonances encountered in flexible rotating structures. These observations complement and reinforce those made in previous, more specific, studies for helicopter rotor blades (Murthy and Hammond, 1981; Hamouda and Pierce, 1984).

The selection of the suspension point a is carried out in the present paper using knowledge of the spatial distribution of the loading $A(\xi)$, and is essentially done by locating the point on the beam at which a zero displacement condition should be imposed that will result in the minimum stress for the given periodic excitation. This, of course, depends on $A(\xi)$, which is typically unknown and varies depending on operating conditions. In such cases, one may use a combination of intelligent guessing, based on the ideas given herein, and trial and error for selection of the position a. In the end, only experimental testing and verification can assure the effectiveness of a proposed design.

There are many extensions possible for the analysis presented herein, and these offer the possibility of refining designs before testing. Of these, damping, nonlinearities, and linear mistuning are briefly discussed here.

Resistive forces on the absorber mass will have a significant effect on the effectiveness of the system. However, such damping is very difficult to predict *a priori* for a given system, and not simple to measure experimentally. Usually the best strategy is to keep damping as small as possible, since the absorber is always optimally tuned (at least to linear order). Also, nonlinearities in the system response can cause severe difficulties if the pendulum amplitude becomes moderately large (Newland, 1964; Sharif-Bakhtiar and Shaw, 1988). This will occur if the mass of the absorber is taken to be too small relative to the loads encountered, and there are almost always spatial constraints or overall weight restrictions that limit the amount of absorber mass that can be used. These nonlinear effects are accounted for in many current designs by using slightly mistuned circular-path absorbers (Newland, 1964). This approach circumvents the nonlinear mistuning by intentionally over-tuning the absorber at linear order so that the absorber comes into proper tuning at moderate amplitudes. While this is effective for avoiding non-linear jump behaviors, it does so at the expense of absorber effectiveness. Along these lines, it should be noted that Hamouda and Pierce (1984) allowed both the pendulum position and its tuning to simultaneously vary, and found that a slightly *under*tuned pendulum offered good performance for suppressing the flapping vibrations of a helicopter rotor blade. More studies on the effects of such intentional mistuning and other absorber paths for rotating beam applications are warranted, especially for specific applications. More interesting recent developments make use of non-circular paths which alleviate the mistuning caused by large motion, but allow the linearized system to remain perfectly tuned (Madden, 1980; Denman, 1992; Lee and Shaw, 1994). Such paths can be implemented by making use of a bifilar configuration for the absorber; see KerWilson (1968) or Denman (1992). Also, the dynamics of absorbers are known to generate higher harmonics, which may be bothersome in flexible structures. The subharmonic absorber system described in Lee and Shaw (1997) may offer a solution to this difficulty.

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