Periodically Forced Linear Oscillator with Impacts: Chaos and Long-Period Motions

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(Received 13 June 1983)

A simple model is discussed for a periodically forced oscillator with a constraint which leads to motions with impacts. For "perfectly plastic" impacts the dynamics is represented by a discontinuous map defined on the circle. The map is shown to undergo period-doubling bifurcations followed by complex sequences of transitions, due to the discontinuities, in which arbitrarily long superstable periodic motions occur.

PACS numbers: 03.20.+i, 02.40.+m

We consider a limiting case of a dynamical problem arising in mechanical systems with amplitude constraints. A single-degree-of-freedom linear oscillator, subject to inertial, sinusoidal excitation, is constrained so that motions are possible only for negative displacement $[x(t)<0]$. When $x(t)=0$, and $\dot{x}(t)\equiv dx(t)/dt>0$, an impact rule[1] is applied:

$$v(t_0^{+})=-\rho u(t_0^{-}). \tag{1}$$

Here u and v are the (relative) velocities of approach and departure, respectively, t_0 is the time of impact, and ρ is the coefficient of restitution. The perfectly elastic case, $\rho=1$, gives rise to a Hamiltonian system and its attendant area-preserving two-dimensional map. In this Letter, in contrast, we consider the perfectly plastic case $\rho=0$, for which a one-dimensional map is obtained. See Holmes[2] and Shaw and Holmes[3,4] for details on mechanical applications and the derivation of equations. This problem represents a simple example in which a one-dimensional map[5] can be derived rigorously in a physically meaningful limit.

The nondimensional equation for motions $x(t)<0$ is

$$\ddot{x}+x=\cos\omega t; \quad x(t_0)=0, \quad \dot{x}(t_0)=y_0; \quad \omega>1, \tag{2}$$

which is solved by

$$x(t;t_0,y_0)=(1-\omega^2)^{-1}\{-\cos\omega t_0\cos(t-t_0)+[\omega\sin\omega t_0+(1-\omega^2)y_0]\sin(t-t_0)+\cos\omega t\}. \tag{3}$$

The first root of the transcendental equation

$$x(t_1;t_0,y_0)=0; \quad t_1>t_0, \tag{4}$$

determines the next impact time t_1, and repeated solutions of (3) and (4), with $y_i=0$ (since $\rho=0$), give an orbit $\{t_i\}_{i=0}^{\infty}$ of the dynamical system. It is more convenient to work with the phase $\varphi_i=t_i \bmod(2\pi/\omega)$ and consider the family of circle maps $f_\omega:S^1\to S^1$ depending on the excitation frequency ω. We will assume $\omega>1$.[3,4] Examples are shown in Fig. 1, below.

The fixed points of f_ω, corresponding to orbits of period n $(t_1-t_0=2\pi n/\omega)$ containing one impact, are easily found from (3) and (4);

$$\varphi_n=\frac{1}{\omega}\arctan\left\{\left[\cos\left(\frac{2\pi n}{\omega}\right)-1\right]\left[\omega\sin\left(\frac{2\pi n}{\omega}\right)\right]^{-1}\right\}; \quad n\leq\left[\frac{\omega+1}{2}\right], \tag{5}$$

where $[x]$ denotes the integer part of x and the restriction is necessary to ensure that "mathematical" orbits do not penetrate the constraint $x=0$.[6] The stability of the fixed points is determined by $f_\omega'(\varphi_n)\equiv(\partial f_\omega/\partial\varphi)_{\varphi=\varphi_n}$,[5] and we find that period-doubling bifurcations occur for $f_\omega'(\varphi_n)=-1$, or

$$2[1-\cos(2\pi n/\omega)]+(1-\omega^2)\sin^2(2\pi n/\omega)=0; \tag{6}$$

this relationship gives bifurcation values $\omega_n\sim 2n+2/\pi$ as $\omega,n\to\infty$. As ω increases and $f_\omega'(\varphi_n)$ decreases through -1, a period-$2n$, two-impact orbit bifurcates from φ_n, but period-doubling cascades[5,7] do not occur here, since the left period-$2n$ point moves into $0\leq\varphi\leq\pi/2\omega$. In the domain $S=[0,\pi/2\omega]\cup[3\pi/2\omega,2\pi/\omega]$, f_ω is flat and has the value $f_\omega(\varphi)\equiv f_\omega(\pi/2\omega)$; thus the period-$2n$ orbit becomes *superstable*.[5] (S corresponds to physical motions in which the oscillating mass adheres to the constraint because of inertial forces until the force changes sign at $\varphi=2\pi/\omega$: Since $\rho=0$, rebounds cannot occur.) Shortly after this, the right period-$2n$ point encounters a discontinuity in the map.

For $2n-1<\omega<2n+1$ f_ω has $n-1$ discontinuities which arise from the fact that orbits in which the oscillating mass just kisses the constraint, $x=0$, separate motions with arbitrarily close initial phase having their first impacts after times $t_1 - t_0 \approx 2\pi n/\omega$ and $t_1' - t_0 \approx 2\pi(n-1)/\omega$, respectively. For $2n-1<\omega<2n+1$ the n connected components of f_ω represent motions in which there are approximately $n, n-1, \ldots, 2, 1$ periods between impacts (reading from left to right). Thus, as ω increases, the left-hand point of the period-$2n$ two-impact orbit remains in S while the right-hand point crosses the discontinuities until we have a superstable orbit containing a point in S and one on the rightmost branch of f_ω, with period $2n-(n-1)=n+1$. This sequence is then repeated.[4]

For the remainder of this Letter we concentrate on the *transitions* in which orbits cross the discontinuities of f_ω. We argue that, while f_ω does not possess a strange attractor or sensitive dependence on initial conditions in the usual sense,[5] its dynamics and bifurcations are nonetheless very complex in this transition region. For simplicity we discuss only the first such region $4.7<\omega<4.9$, following the period-doubling bifurcation at $\omega_2 = 4.6572$. A sequence of maps f_ω for this region is shown in Fig. 1. In this range there are two unstable fixed points, φ_2 and φ_1, marked L and R, corresponding to period-2 and period-1 orbits, respectively.

We start with a period-4, two-impact orbit [Fig. 1(a)]. Directly after crossing the discontinuity, the orbit contains three points, 1 on the period-2 branch and 2 and 3 on the period-1 branch. Thus it still has period 4 but now contains three impacts [Fig. 1(b)]. As ω increases, 2 moves rightward and 3 leftward, and continuous dependence on ω implies that there are values

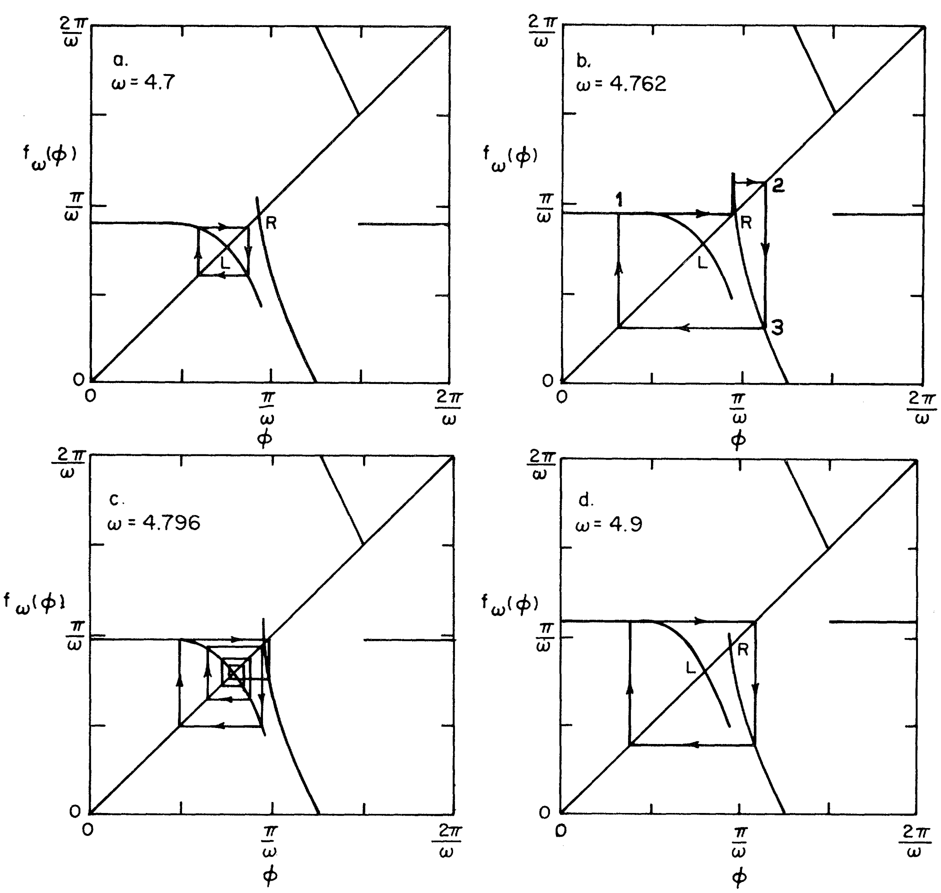

FIG. 1. One-dimensional (forcing phase) maps $f_\omega : S^1 \to S^1$ arising from the impact oscillator. (a) $\omega = 4.7$: period-4, two-impact stable orbit. (b) $\omega = 4.762$: period-4, three-impact superstable orbit. (c) $\omega = 4.796$: period-15, eight-impact superstable orbit occurring close to ω'', at which $f_\omega^{\,2}(S) = L$. (d) $\omega = 4.9$: period-3, two-impact superstable orbit.

$\omega'' < \omega' \in (4.7, 4.9)$ for which $f_{\omega''}^{\,2}(S) = L$ and $f_{\omega'}(S) = R$. Such orbits are analogous to those occurring at the countable set of "Misiurewicz points" for C^3 one-dimensional maps with a single critical point c, in which some iterate $f^m(c)$ lands on an unstable periodic point. In that case the maps are known to have strange attractors on a finite union of intervals which support an absolutely continuous invariant measure.[5,8] Here, in contrast, almost all points (including those in S) are mapped to a single point (L or R) and thus the measure has only point support. However, these bifurcation values do play a role analogous to the corresponding ones in continuous maps in that they are accumulation points for parameter intervals over which arbitrarily long periodic orbits exist; cf. Ref. 9. Such orbits are easily constructed by reference to Fig. 1(c), for example. One selects a parameter value such that $f_\omega(S)$ lies arbitrarily close to L (above *or* below), in which case successive iterates spiral away until one lands in S. We note that the accumulation rate of these intervals is not universal, but depends primarily on the derivative $f_\omega'(\varphi_2)$ of the map at L. The process can be iterated to yield periodic orbits spending arbitrarily long times near L and then R in irregular sequences of "period-2" and "period-1" jumps. Whenever they contain a point in S such orbits are superstable. In fact following ω'' and accumulating upon ω' from below are countably many "homoclinic" parameter values for which $f_\omega^{\,2n}(S) = L$ and accumulating from above values for which $f_\omega^{\,2n+1}(S) = L$. These terminate with $f_\omega(S) = R$, after which $f_\omega(S)$ moves down the period-1 branch until we have the simple orbit of Fig. 1(d). As for continuous one-dimensional maps, we can iterate this procedure to produce a self-similar bifurcation diagram containing nested or "box-within-box" structures.[10]

Note that, while at the homoclinic bifurcation values L (or R) attracts a set of nonzero measure (for some values it attracts almost all points), it is not an attractor in the usual sense, since orbits starting in any neighborhood U of L leave U before eventually returning. Such "attractors" are extremely sensitive to small perturbations (in ω), but do not display sensitive dependence on initial conditions, since the flat region of f_ω over S contracts whole intervals of initial data.

We can summarize the gross aspects of the dynamics of f_ω in the bifurcation diagram of Fig. 2, which shows the successions of period-doubling bifurcations followed by transitions in which

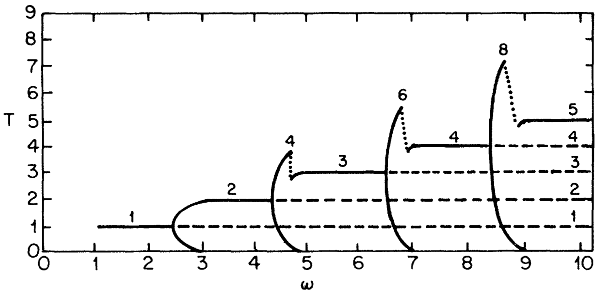

FIG. 2. A bifurcation diagram summarizing the low-period stable motions. Solid line, stable; dashed line, unstable; dotted line, transition region. Ordinate indicates period between impacts in multiples of $T = 2\pi/\omega$; number above branch also indicates period.

the period is reduced from $2n$ to $n+1$ as a result of passage over the $n-1$ discontinuities. We expect the dynamics within transitions for $n \geqslant 3$ to be at least as complex as that for $n = 2$, considered above.

For slightly higher values of ω ($\omega \approx 5$) it is possible to prove[4] that, along with the period-3 superstable orbit containing a point on each branch of f_ω, there is also an invariant Cantor set C supported on two disjoint subintervals I_2 (containing L) and I_1 (containing R). The dynamics of f_ω, restricted to C, is conjugate to a shift on two symbols.[11] Thus, orbits visiting I_1 and I_2 in *any* preassigned sequence can be found simultaneously for the *same* value of ω, including uncountably many nonperiodic motions and an orbit dense in C. All these orbits are unstable and hence correspond to transient chaos or "preturbulence." The set C can be regarded as the ghost of the set of arbitrarily long, stable, periodic motions created during the transition region. In a similar manner, after the last attractor vanishes at $\mu = 2$ in the one-dimensional family $x \to \mu - x^2$, a shift on two symbols remains.

We close by remarking that simple implicit-function–theorem arguments permit many of these results to be generalized to the case of large but finite dissipation at impacts ($0 < \rho \ll 1$). In particular, the two-shift for $\omega \approx 5.0$ still exists[4] and can be proved hyperbolic,[12] and any (super)-stable orbit of period n occurring in the transition region will persist in a nearby ω interval for ρ (depending on n) sufficiently small; however, $\rho(n)$ may approach 0 as $n \to \infty$.

[1]J. P. Meriam, *Dynamics* (Wiley, New York, 1975).
[2]P. J. Holmes, J. Sound Vib. 84, 173 (1982).

[3]S. W. Shaw and P. J. Holmes, to be published.
[4]S. W. Shaw and P. J. Holmes, to be published.
[5]P. Collet and J. P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Birkhäuser, Boston, 1980).
[6]M. Senator, J. Acoust. Soc. Am. 47, 1390 (1970).
[7]M. J. Feigenbaum, J. Stat. Phys. 19, 25 (1978).
[8]M. Misiurewicz, Inst. Hautes Etudes Sci. Publ. Math. 53, 17 (1981).
[9]F. Marotto, J. Math. Anal. Appl. 63, 199 (1978).
[10]I. Gumowski and L. Mira, *Recurrences and Discrete Dynamical Systems*, Lecture Notes in Mathematics No. 809, (Springer-Verlag, New York, 1980).
[11]J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer, New York, 1983).
[12]S. W. Shaw, Ph.D. thesis, Cornell University, 1983 (unpublished).