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ORDER-TUNED VIBRATION ABSORBERS FOR CYCLIC ROTATING FLEXIBLE STRUCTURES

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ABSTRACT

This paper investigates the use of order-tuned absorbers to attenuate vibrations of flexible blades in a bladed disk assembly subjected to engine order excitation. The blades are modeled by a cyclic chain of N oscillators, and a single vibration absorber is fitted to each blade. These absorbers exploit the centrifugal field arising from rotation so that they are tuned to a given order of rotation, rather than to a fixed frequency. A standard change of coordinates based on the cyclic symmetry of the system essentially decouples the governing equations of motion, yielding a closed form solution for the steady-state response of the overall system. These results show that optimal reduction of blade vibrations is achieved by tuning the absorbers to the excitation order n , but that the resulting system is highly sensitive to small perturbations. Intentional *detuning* (meaning that the absorbers are slightly over- or under-tuned relative to n) can be implemented to improve the robustness of the design. It is shown that by slightly undertuning the absorbers there are no system resonances near the excitation order of interest and that the resulting system is robust to *mistuning* (i.e., small random uncertainties in the system parameters) of the absorbers and/or blades. These results offer a basic understanding of the dynamics of a bladed disk assembly fitted with order-tuned vibration absorbers, and serve as a first step to the investigation of more realistic models, where, for example, imperfections and nonlinear effects are considered, and multi-DOF and general-path absorbers are employed.

INTRODUCTION

This work investigates the use of order-tuned absorbers to attenuate vibrations in a system with cyclic symmetry. The applications of interest are rotating flexible structures, and in particular turbine blades, bladed disk assemblies, and blisks (integral disk-blade systems). These systems are nominally cyclic and are subjected to traveling wave dynamic loading—the so-called *engine order excitation*—which is characterized by excitation frequencies that are proportional to the mean rotational speed of the rotor. Such excitations can lead to high cycle fatigue (HCF) failure, noise, reduced performance, and other undesirable effects. This is an ideal setting for the use of centrifugally-driven, order-tuned vibration absorbers, yet their implementation to such systems has received little attention to date. An important feature of these absorber systems is that they are tuned to a particular order of rotation, and are thus effective over a range of operating speeds, rather than to a particular speed. This is in contrast to the classical *frequency-tuned* absorbers due to Ormondroyd and Den Hartog [1, 2], which are effective only at a particular frequency. The desired *order-tuning* [3] of the proposed absorbers is achieved by exploiting the centrifugal field to provide the absorber restoring forces, as opposed to a spring element. Much is already known about the dynamic behavior of systems of vibration absorbers, and the same is true for systems with symmetries in general and for rotating flexible structures in particular. This work aims to apply the theory and methodology of order-tuned absorbers to such systems, an effort that is partly motivated by the recent experimental work of Duffy *et al* [4].

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The model considered here fits into a class of periodic structures whose geometry and structural properties are rotationally periodic, and is said to have *cyclic symmetry* [5]. It consists of N sectors (Fig. 1b), each of which is composed of a single blade/absorber combination. Hence each sector possesses two degrees of freedom (DOF), and these are coupled such that the overall system (Fig. 1a) has $2N$ DOF. While it is possible to handle the governing (coupled) $2N$ equations of motion using the tools from elementary vibration theory, a standard change of coordinates based on the cyclic symmetry of the system is employed to reduce the problem to N uncoupled 2-DOF systems. Not only does this offer substantial computational savings, it also yields closed-form solutions describing the forced response of each sector and it clearly shows the effects of the absorbers on the system. The main objective of this work is to investigate the overall system behavior and to assess the effectiveness of the absorbers in attenuating the blade vibrations, especially near resonance. Absorber tuning plays a key role in this analysis.

The cyclicity of the bladed disk model in Fig. 1a results in block circulant system matrices. It is not supposed that the reader has encountered such matrices before, and hence their basic properties are outlined in the appendix along with other relevant mathematical preliminaries. A detailed account of the theory of circulants can be found in the classical text by Davis [6].

The paper is organized as follows. The system model is described in the next section and the dimensionless equations of motion are hence formulated for a single sector and subsequently for the overall system. The forced response of the linear system is then considered, wherein the effectiveness of the absorbers is investigated and a particular absorber tuning strategy is motivated. Finally, the paper closes with some concluding remarks and directions for future work.

MATHEMATICAL MODEL

Consider the model for a bladed disk shown schematically in Fig. 1a. It consists of a rigid rotor of radius R , which rotates about a fixed axis at O with a constant angular speed Ω . A total of N identical blades (each modeled by a simple pendulum of length L and mass M), referred to as the primary systems, are uniformly attached around the circumference of the rotor. The single-mode flexural stiffness of each blade is captured by identical linear torsional springs of stiffness k_b , and the inter-blade elastic coupling (due to shrouds, disk flexibility, etc.) is captured by linear springs of stiffness k_c . The coupling springs connect blade i to blade $i + 1$ a distance b radially along the pendulums relative to their attachment points to the rotor. All springs are unstressed when the blades are in a purely radial configuration, that is, when $\theta_i = 0$, ($i \in \mathcal{N}$), where $\mathcal{N} = \{1, 2, \dots, N\}$ is the set of blade numbers. A vibration absorber of mass m (typically $m \ll M$) is fitted to each blade. The absorbers ride on circular paths of radius r , which are centered a distance αL ($\alpha \leq 1$) radi-

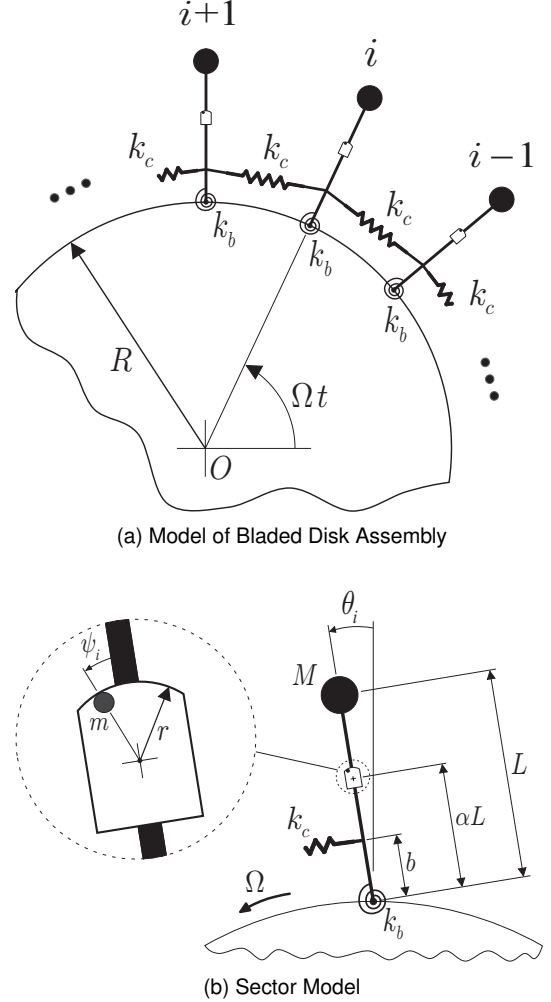


Figure 1. (a) Model of bladed disk assembly and (b) sector model.

ally along the blade pendulums. Their dynamics are captured by the angular coordinates ψ_i ($i \in \mathcal{N}$), which are measured relative to the blades. Physically, the absorber amplitudes $|\psi_i|$ are limited to some value ψ_o by stops, which represent the rattling space limits imposed by the geometry of the turbine blades. Impacts occur whenever $|\psi_i| = \psi_o$, the dynamics of which are investigated analytically in [7] for the case of a single isolated blade/absorber combination, and experimentally in [4]. In the present analysis it is assumed that θ_i and $|\psi_i| < \psi_o$ are sufficiently small so that the dynamics of the overall system are captured by a linearized model.

The primary systems (blades) are harmonically excited in the transverse sense by tip forces

$$F_i(t) = F_o e^{j\theta_i} e^{jn\Omega t}, \quad i \in \mathcal{N} \quad (1)$$

where F_o is the strength of the excitation, $j = \sqrt{-1}$, t is time, and

$$\phi_i = 2\pi \frac{n}{N}(i-1), \quad i \in \mathcal{N} \quad (2)$$

are inter-blade phase angles. Equation (1) is known as *engine order excitation* and n is the *engine order*. In what follows, we restrict $0 < n < N$ to be an integer, which is typically the case in turbomachinery applications. The case of noninteger n is investigated in [8].

Absorber, blade, and inter-blade damping is included in the model (not reflected in Fig. 1) and is captured by the linear torsional and translational dampers with constants c_a , c_b , and c_c .

EQUATIONS OF MOTION

The equations of motion for each 2-DOF sector are derived by employing Lagrange's method and are linearized for small motions of the primary and absorber systems, that is, for small θ_i and ψ_i . The resulting linear equations are divided through by the inertia term ML^2 and time is rescaled according to $\tau = \omega_o t$, where

$$\omega_o = \sqrt{(k_b/L^2)/M} \quad (3)$$

is the undamped natural frequency of a single isolated blade (with no absorber) with $k_c = 0$ and $\Omega = 0$. Then the dynamics of the i^{th} sector are governed by

$$\left. \begin{aligned} \mathbf{M}\mathbf{z}_i'' + \mathbf{C}\mathbf{z}_i' + \mathbf{K}\mathbf{z}_i + \mathbf{C}_c(-\mathbf{z}_{i-1}' + 2\mathbf{z}_i' - \mathbf{z}_{i+1}') \\ + \mathbf{K}_c(-\mathbf{z}_{i-1} + 2\mathbf{z}_i - \mathbf{z}_{i+1}) \\ = \mathbf{f}e^{j\phi_i}e^{jn\sigma\tau} \end{aligned} \right\}, \quad i \in \mathcal{N} \quad (4)$$

where

$$\mathbf{z}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} \theta_i/\Psi_o \\ \psi_i/\Psi_o \end{bmatrix} \quad (5)$$

is a vector of nondimensional physical coordinates (x_i and y_i describe the dynamics of the i^{th} blade and absorber, respectively), $\sigma = \Omega/\omega_o$ is the dimensionless angular speed of the rotor, and $(\cdot)' = d(\cdot)/d\tau$. In Eq. (4) the subscripts are taken mod N such that $\mathbf{z}_{N+1} = \mathbf{z}_1$ and $\mathbf{z}_0 = \mathbf{z}_N$. The elements of the sector mass, damping, and stiffness matrices are given in Table 2, and their attendant parameters are defined in Table 1. The matrices

$$\mathbf{C}_c = \begin{bmatrix} \xi_c & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{K}_c = \begin{bmatrix} v^2 & 0 \\ 0 & 0 \end{bmatrix},$$

Table 1. Selected list of dimensionless parameters.

Parameter	Description
$f = F_o L / k_b \Psi_o$	Strength of the engine order excitation
α	Distance from blade base to absorber base point
$\beta = (\bar{n} - n) / n$	Absorber detuning
$\gamma = r / L$	Effective length of the absorber pendulum
$\delta = R / L$	Radius of the rotor disk
$\mu = m / M$	Absorber-to-blade mass ratio
$v = \omega_c / \omega_o = \sqrt{\frac{k_c}{k_b / b^2}}$	Inter-blade coupling strength
$\xi_a = \frac{c_a / L^2}{\sqrt{(k_b / L^2) M}}$	Torsional damping constant for the absorber
$\xi_b = \frac{c_b / L^2}{\sqrt{(k_b / L^2) M}}$	Torsional damping constant for the blade
$\xi_c = \left(\frac{b}{L}\right)^2 \frac{c_c}{\sqrt{(k_b / L^2) M}}$	Coupling damping constant
$\sigma = \Omega / \omega_o$	Angular speed of the rotor

Table 2. Elements of \mathbf{M} , \mathbf{C} , and \mathbf{K} .

$M_{11} = 1 + \mu(\alpha + \gamma)^2$	$C_{11} = \xi_b$	$K_{11} = 1 + (1 + \mu(\alpha + \gamma))\delta\sigma^2$
$M_{12} = \mu\gamma(\alpha + \gamma)$	$C_{12} = -\xi_a$	$K_{12} = \mu\gamma\delta\sigma^2$
$M_{21} = M_{12}$	$C_{21} = 0$	$K_{21} = K_{12}$
$M_{22} = \mu\gamma^2$	$C_{22} = \xi_a$	$K_{22} = \mu\gamma(\alpha + \delta)\sigma^2$

capture the inter-blade coupling and vanish when $\xi_c = 0$ and $v = 0$, respectively, in which case Eq. (4) describes the forced motion of N isolated blade/absorber systems. (Equation (4) is studied in detail in [7] for the case when $N = 1$, $\mathbf{K}_c = \mathbf{0}$, and $\mathbf{C}_c = \mathbf{0}$, including the impact dynamics that occur when $y_i = 1$, that is, when $\psi_i = \Psi_o$.) The sector forcing vector is given by

$$\mathbf{f} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad (6)$$

where f is defined in Table 1. Finally, the parameter $\omega_c = \frac{b}{L}\sqrt{k_c/M}$ (see v in Table 1) is the undamped natural frequency of a single isolated blade (with no absorber) with $k_b = 0$ and $\Omega = 0$ and with a single coupling stiffness element k_c connected to an adjacent, stationary blade.

By stacking the N vectors \mathbf{z}_i into the system configuration vector $\mathbf{q} = (\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_N)^T$, the governing equations of motion for the overall $2N$ -DOF system take the form

$$\hat{\mathbf{M}}\mathbf{q}'' + \hat{\mathbf{C}}\mathbf{q}' + \hat{\mathbf{K}}\mathbf{q} = \hat{\mathbf{f}}e^{jn\sigma\tau}, \quad (7)$$

where $\hat{\mathbf{M}}$ is block diagonal with diagonal blocks \mathbf{M} and $\hat{\mathbf{K}}$ is block circulant with the N generating matrices $\mathbf{K} + 2\mathbf{K}_c, -\mathbf{K}_c, \mathbf{0}, \dots, \mathbf{0}, -\mathbf{K}_c$ in its first row (and column). (See [6] for a comprehensive treatment of circulant matrices and their properties. A brief review of such matrices is given in the appendix.) The matrix $\hat{\mathbf{C}}$ is similarly defined by replacing \mathbf{K} with \mathbf{C} and \mathbf{K}_c with \mathbf{C}_c in $\hat{\mathbf{K}}$. In terms of the circulant operator, the system mass, damping, and stiffness matrices are given by

$$\left. \begin{aligned} \hat{\mathbf{M}} &= \text{circ}(\mathbf{M}, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, \mathbf{0}) = \text{diag}(\mathbf{M}) \\ \hat{\mathbf{C}} &= \text{circ}(\mathbf{C} + 2\mathbf{C}_c, -\mathbf{C}_c, \mathbf{0}, \dots, \mathbf{0}, -\mathbf{C}_c) \\ \hat{\mathbf{K}} &= \text{circ}(\mathbf{K} + 2\mathbf{K}_c, -\mathbf{K}_c, \mathbf{0}, \dots, \mathbf{0}, -\mathbf{K}_c) \end{aligned} \right\}, \quad (8)$$

where the $\text{circ}(\cdot)$ operation is defined in the appendix. Finally, the system forcing vector is

$$\hat{\mathbf{f}} = (\mathbf{f}e^{j\phi_1}, \mathbf{f}e^{j\phi_2}, \dots, \mathbf{f}e^{j\phi_N})^T, \quad (9)$$

where \mathbf{f} is given by Eq. (6) and the i^{th} inter-blade phase angle ϕ_i is defined by Eq. (2).

FORCED RESPONSE

Equation (7) can be handled using standard techniques, and its solution in the steady-state is given by

$$\mathbf{q}^{ss}(\tau) = \hat{\mathbf{Z}}^{-1} \hat{\mathbf{f}} e^{jn\sigma\tau}, \quad (10)$$

where $\hat{\mathbf{Z}} = \hat{\mathbf{K}} - n^2\sigma^2\hat{\mathbf{M}} + jn\sigma\hat{\mathbf{C}}$ is the system impedance matrix. This requires computation of $\hat{\mathbf{Z}}^{-1}$, however, which can be quite involved for any reasonable bladed disk model. It is known that, due to its cyclic symmetry, Eq. (7) can be decoupled via a modal transformation to a set of N reduced-order models, each with 2 DOF. (The reduced-order models have the same number of DOF as an individual sector.) A special feature of these decoupled systems is that only the mode $n+1$ is excited, provided that the engine order $0 < n < N$ is an integer (as shown subsequently). Hence the steady-state response of the overall $2N$ -DOF system reduces to the solution of a single, harmonically forced, 2-DOF system.

A special case is first considered in which the blades are locked in their zero positions relative to the rotor. This motivates an absorber tuning parameter, which is employed to tune the absorbers (close) to a given order. The forced response of the general system, in which the absorbers are free to move, is subsequently detailed.

Response with the Blades Locked

As a special case we consider the forced response with the blades locked in their zero positions. This leads to a system of dynamically isolated absorbers that oscillate freely under the influence of centrifugal effects. These dynamics follow from Eq. (4) by setting $x_i = x'_i = x''_i \equiv 0$, and are governed by

$$M_{22}y''_i + C_{22}y'_i + K_{22}y_i = 0, \quad i \in \mathcal{N} \quad (11)$$

where the mass, damping, and stiffness terms M_{22} , C_{22} , and K_{22} are defined in Table 2. Equation (11) is a set of N uncoupled and unforced single-DOF harmonic oscillators. The dimensionless undamped natural frequency of their free vibration is given by

$$\bar{\omega}_{22} \equiv \frac{\omega_{22}}{\omega_o} = \sqrt{\frac{\alpha + \delta}{\gamma}} \sigma \equiv \tilde{n}\sigma, \quad (12)$$

or $\omega_{22} = \tilde{n}\Omega$ in dimensional form, where ω_o is given by Eq. (3) and

$$\tilde{n} = \sqrt{\frac{\alpha + \delta}{\gamma}} \quad (13)$$

is defined to be the *absorber tuning order*. Since the absorbers are restrained only through centrifugal effects, $\bar{\omega}_{22}$ scales directly with σ [4, 9]. This feature is exploited to tune the absorbers to a given *order* of the excitation, rather than to a fixed *frequency*, as is done in the classical sense [2]. The tuning parameter \tilde{n} is used for this purpose and is determined by selecting the dimensionless curvature of the pendulum absorber γ (dimensionally r) and the distance of its effective attachment point from the center of rotation of the rotor, that is, $\alpha + \delta$ (dimensionally $\alpha L + R$).

Response of the General System

We now turn to the dynamics of the overall $2N$ -DOF system, which are governed by Eq. (7). Since the system matrices $\hat{\mathbf{M}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{K}}$ are block circulant they can be block diagonalized via a unitary (similarity) transformation, the form of which is given by Eq. (38) of the appendix. Such a diagonalization is accomplished by introducing the change of coordinates¹

$$\mathbf{q} = (\mathbf{F} \otimes \mathbf{I})\mathbf{u}, \quad \text{or} \quad \mathbf{z}_i = (\mathbf{e}_i^T \otimes \mathbf{I})\mathbf{u}, \quad i \in \mathcal{N} \quad (14)$$

¹The reader who is not familiar with transformations of this type should regard Eq. (14) as the usual modal transformation employed in elementary linear vibration theory.

where \mathbf{F} is the $N \times N$ complex Fourier matrix and \mathbf{e}_i is its i^{th} column, \otimes denotes the Kronecker product (these are defined in the appendix), \mathbf{I} is the 2×2 identity matrix (the order of \mathbf{I} corresponds to the number of DOF in each sector), and $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N)^T$ is a vector of *cyclic coordinates* with each \mathbf{u}_i of dimension 2×1 . Substituting Eq. (14) into Eq. (7) and multiplying from the left by the unitary matrix $(\mathbf{F} \otimes \mathbf{I})^{\mathcal{H}} = (\mathbf{F}^{\mathcal{H}} \otimes \mathbf{I})$ yields a system of N block decoupled equations. They are

$$\tilde{\mathbf{M}}_p \mathbf{u}_p'' + \tilde{\mathbf{C}}_p \mathbf{u}_p' + \tilde{\mathbf{K}}_p \mathbf{u}_p = (\mathbf{e}_p^{\mathcal{H}} \otimes \mathbf{I}) \hat{\mathbf{f}} e^{jn\sigma\tau}, \quad p \in \mathcal{N} \quad (15)$$

where $(\cdot)^{\mathcal{H}} = (\bar{\cdot})^T$ denotes the conjugate transpose. The 2×2 mass, damping, and stiffness matrices associated with the p^{th} mode are given by

$$\left. \begin{aligned} \tilde{\mathbf{M}}_p &= \mathbf{M} \\ \tilde{\mathbf{C}}_p &= \mathbf{C} + 2\mathbf{C}_c \left(1 - \cos\left(\frac{2\pi(p-1)}{N}\right)\right) \\ \tilde{\mathbf{K}}_p &= \mathbf{K} + 2\mathbf{K}_c \left(1 - \cos\left(\frac{2\pi(p-1)}{N}\right)\right) \end{aligned} \right\}, \quad p \in \mathcal{N} \quad (16)$$

which follow from Eq. (39) of the appendix.

It is assumed in this work that $0 < n < N$ is an integer. Hence the p^{th} modal forcing vector simplifies considerably, and is given by

$$(\mathbf{e}_p^{\mathcal{H}} \otimes \mathbf{I}) \hat{\mathbf{f}} = \begin{cases} \sqrt{N} \mathbf{f}, & p = n + 1 \\ \mathbf{0}, & \text{otherwise} \end{cases} \quad (17)$$

where $\mathbf{0} = (0, 0)^T$. Equation (17) shows that only mode $n + 1$ is excited and, therefore, $\mathbf{u}_{n+1}(\tau)$ is the only non-zero modal response in the steady-state.²

Assuming harmonic motion, the p^{th} steady-state modal response follows easily from Eq. (15) is given by

$$\mathbf{u}_p^{ss}(\tau) = \begin{cases} \sqrt{N} \tilde{\mathbf{Z}}_{n+1}^{-1} \mathbf{f} e^{jn\sigma\tau}, & p = n + 1 \\ \mathbf{0}, & \text{otherwise} \end{cases} \quad (18)$$

where

$$\tilde{\mathbf{Z}}_p = \tilde{\mathbf{K}}_p - n^2 \sigma^2 \tilde{\mathbf{M}}_p + jn\sigma \tilde{\mathbf{C}}_p \quad (19)$$

is the p^{th} modal impedance matrix. The response of sector i (in physical coordinates) follows from the transformation given by Eq. (14) and is given by $\mathbf{z}_i^{ss} = (\mathbf{e}_i^T \otimes \mathbf{I}) \mathbf{u}^{ss}$, where $\mathbf{u}^{ss}(\tau) = (\mathbf{0}, \dots, \mathbf{0}, \mathbf{u}_{n+1}^{ss}(\tau), \mathbf{0}, \dots, \mathbf{0})^T$. In terms of the 2×2 modal impedance matrix $\tilde{\mathbf{Z}}_p$ and the sector forcing vector \mathbf{f} , the response of the i^{th} sector is

$$\mathbf{z}_i^{ss}(\tau) = \tilde{\mathbf{Z}}_{n+1}^{-1} \mathbf{f} e^{j\phi_i} e^{jn\sigma\tau}, \quad i \in \mathcal{N}. \quad (20)$$

Equation (20) shows that each blade/absorber combination behaves identically except for a constant phase shift from one sector to another, which is captured by the inter-blade phase angle ϕ_i . This approach offers a significant computational advantage over the brute-force solution to the full $2N$ -DOF system, as given by Eq. (10).

The $2N$ natural frequencies of the system are defined implicitly by $\det(\hat{\mathbf{K}} - \bar{\omega}^2 \hat{\mathbf{M}}) = 0$, the solution of which can be quite involved for any reasonable bladed disk model. This effort can be significantly reduced, however, by instead using the N modal matrices given by Eq. (16). We recall that each $\tilde{\mathbf{M}}_p$ and $\tilde{\mathbf{K}}_p$ follow from a unitary (similarity) transformation of $\hat{\mathbf{M}}$ and $\hat{\mathbf{K}}$ and hence the system natural frequencies are preserved. They follow from the N , second-order characteristic polynomials

$$\det(\tilde{\mathbf{K}}_p - \bar{\omega}^2 \tilde{\mathbf{M}}_p) = 0, \quad p \in \mathcal{N}. \quad (21)$$

By inspection of each modal stiffness matrix $\tilde{\mathbf{K}}_p$ it is clear that there will be repeated natural frequencies due to the presence of the harmonic term $\cos(\frac{2\pi}{N}(p-1))$. This degeneracy in the eigenvalue problem is due to the block circulant, block symmetric structure of $\hat{\mathbf{K}}$. The natural frequencies corresponding to $p = 1$ (zero harmonic) are distinct, but the remaining eigenvalues appear in repeated pairs, except for the case of even N , in which case the $p = (N+2)/2$ frequency ($N/2$ harmonic) is also distinct.³

The dimensionless natural frequencies $\bar{\omega}_{1,2}^{(i)}$ ($i \in \mathcal{N}$) are plotted in Fig. 2 in terms of the rotor speed σ for $N = 10$, $n = 2$, for a particular sector model, and for various levels of inter-blade coupling v . In these Campbell diagrams, the $N = 10$ natural frequencies $\bar{\omega}_1^{(i)}$ branching from $\sigma = 0$ (in this case there are four repeated pairs) correspond to in-phase modes, wherein the absorber/blade combination in each sector oscillates in phase. The remaining N natural frequencies $\bar{\omega}_2^{(i)}$ correspond to the out-of-phase modes. As is shown in Fig. 2a, the frequencies $\bar{\omega}_1^{(i)}$ and also $\bar{\omega}_2^{(i)}$ ($i \in \mathcal{N}$) are nearly coincident when the inter-blade coupling is weak, that

²There are multiple modes excited in the general case when n is not an integer. Also, it is a simple matter to include $n \geq N$, and one finds that an engine order n excites mode $p = n \bmod N + 1$. These situations are considered in [8].

³The eigenvalue corresponding to the zero harmonic of a symmetric circulant matrix is *always* distinct. However, there are M such eigenvalues for block symmetric, block circulant matrices with $M \times M$ blocks, and these may be repeated. The same is true for eigenvalues corresponding to the $N/2$ harmonic.

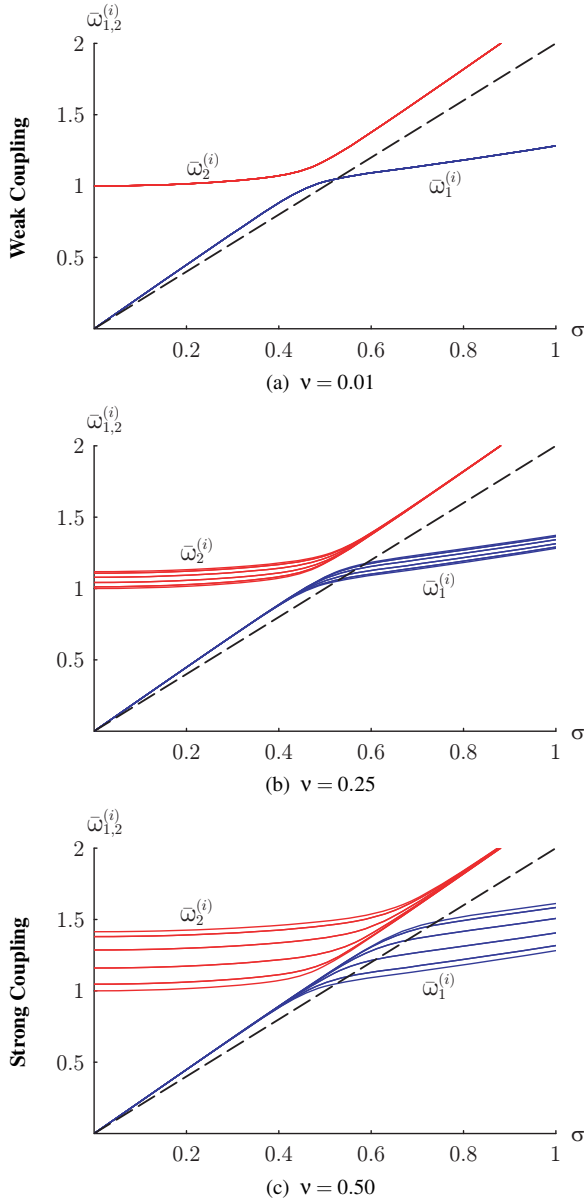


Figure 2. Campbell diagrams showing the dimensionless natural frequencies $\bar{\omega}_{1,2}^{(i)}$ (—) versus the dimensionless rotor speed σ for $N = 10$, $n = 2$, $\alpha = 0.84$, $\gamma = 0.30$, $\delta = 0.67$, and $\mu = 0.015$: (a) $\nu = 0.01$; (b) $\nu = 0.25$; (c) $\nu = 0.50$. (---) denotes the engine order line $n\sigma$.

is, when ν is small (in the absence of inter-blade coupling, they are identically coincident and there are exactly N repeated natural frequencies) and they spread out for increasing ν . This is shown in Fig. 2b and Fig. 2c.

In general, there may be a system resonance whenever

$$n\sigma = \bar{\omega}_{1,2}^{(i)}(\sigma), \quad \text{or} \quad n\Omega = \omega_{1,2}^{(i)}(\Omega), \quad (22)$$

(here, $\bar{\omega}_{1,2}^{(i)} = \omega_{1,2}^{(i)}/\omega_o$) and these possible resonances can be identified in Fig. 2 by the intersections of the natural frequency curves with the order line $n\sigma$ (shown as dashed lines). The main objective of this work is to select the absorber parameters to avoid such resonances over a range of rotor operating speeds, which is the subject of the next section.

As a final note, the inter-blade coupling ν is generally quite small, typically on the order of 1%. However, in order to show clearly which modes are excited and the effects of absorber (de)tuning,⁴ a rather large (unrealistic) value for the coupling will be employed in the ensuing numerical analysis; this does not qualitatively affect the approach nor the conclusions.

Absorber Tuning

Absorber tuning refers to a particular choice of absorber parameters to attenuate, as much as possible, the response of the primary systems (blades) over a range of operating speeds. This is done by prescribing the dimensionless parameters μ , γ , and α , which in turn specifies the absorber mass m , the radius of its path r , and its placement along the blades, respectively. It is shown subsequently that, in the absence of damping, there exists a particular absorber tuning such that full annihilation of the blade vibrations is possible at the expense of localized vibrations of the absorbers. This is done quite simply by setting the natural frequency of the isolated (undamped) absorbers to that of the excitation, just as is done with the classical dynamic vibration absorber [2] and also the centrifugal pendulum vibration absorber [3]. In the presence of small absorber damping, however, it becomes impossible to eliminate the blade vibrations completely. In what follows we develop an absorber tuning strategy for the undamped case. The effects of damping on the absorber performance is investigated in [10].

It is customary to introduce the tuning order \tilde{n} as one of the absorber parameters, and this is done in the present study via the substitution

$$\gamma = \frac{\alpha + \delta}{\tilde{n}^2}, \quad (23)$$

which follows from Eq. (13). Then, in the absence of damping and after some simplification, Eq. (20) reduces to

$$\begin{bmatrix} x_i^{ss}(\tau) \\ y_i^{ss}(\tau) \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} e^{j\phi_i} e^{jn\sigma\tau}, \quad i \in \mathcal{N} \quad (24)$$

⁴In this work *detuning* means that the absorbers are slightly over- or under-tuned relative to n . This is not to be confused with *mistuning*, which refers to small random uncertainties in the system parameters. In the turbomachinery literature, detuning and mistuning are often used interchangeably, but we need to make a clear distinction between them in this investigation.

where

$$\left. \begin{aligned} X &= \frac{f\tilde{n}^2(n^2 - \tilde{n}^2)}{\Gamma} \\ Y &= -\frac{f\tilde{n}^2(\delta(n^2 - \tilde{n}^2) + \alpha(1 + \tilde{n}^2)n^2)}{(\alpha + \delta)\Gamma} \end{aligned} \right\} \quad (25)$$

are the blade and absorber steady-state response amplitudes and

$$\begin{aligned} \Gamma &= \mu\alpha^2(1 + \tilde{n}^2)^2n^2\sigma^2 + \tilde{n}^2(n^2 - \tilde{n}^2) \\ &\quad + 2\nu^2\tilde{n}^2(n^2 - \tilde{n}^2) \left(1 - \cos\left(\frac{2\pi n}{N}\right)\right) \\ &\quad + (n^2 - \tilde{n}^2)(\tilde{n}^2(\delta - n^2) + \mu\alpha\delta(1 + \tilde{n}^2))\sigma^2. \end{aligned}$$

The ideal absorber tuning follows by inspection of the first entry of Eq. (25), and is given by

$$\tilde{n} = n. \quad (26)$$

If the system is tuned according to Eq. (26) the blade and absorber amplitudes reduce to

$$\left. \begin{aligned} X &= 0 \\ Y &= -\frac{fn^2}{\mu\alpha(\alpha + \delta)(1 + n^2)\sigma^2} \end{aligned} \right\}, \quad (\tilde{n} = n) \quad (27)$$

which shows that the blade vibrations can be eliminated completely. In the presence of small damping the tuning strategy given by Eq. (26) allows for a significant reduction of X , but a complete elimination of the blade amplitudes becomes impossible. (This is investigated in [10].) The absorber amplitudes given by Y are inversely proportional to the mass ratio μ and also to $\alpha(\alpha + \delta)$. It is therefore desirable to make the absorber masses large relative to the blade mass and place them as close to the end of the blade as possible. In practice, however, there are limits on the size and makeup of the absorber masses (typically μ is very small, on the order of 10^{-2} to 10^{-3}) and their placement relative to the blades. The negative sign in Y implies that the absorbers oscillate out of phase with respect to the engine order excitation, which counters its action on the blades.

By implementing the absorber tuning given by Eq. (26) one is simply setting the natural frequency of the isolated absorbers to the excitation frequency, that is, $\tilde{n}\sigma = n\sigma$, and the absorbers are said to be *exactly tuned*. Again, we emphasize that the said tuning is valid at all rotation speeds, a feature that is made possible by the structure of $\tilde{\omega}_{22} = \tilde{n}\sigma$. However, any perturbation

of the model or absorber parameters, due to in-service wear, environmental effects, and so on, invariably destroys the absorber tuning. To see this, and to allow for intentionally detuned designs, we let

$$\tilde{n} = n(1 + \beta), \quad (28)$$

where β is a detuning parameter. Figure 3 shows frequency response curves for an undamped system tuned according to Eq. (28) with $N = 10$, $n = 3$, $f = 0.01$, for a particular sector model, and for various levels of absorber detuning β . Note that any significant level of detuning results in a system resonance of mode $n + 1$, which is shown in Fig. 3a for $\beta = -0.10$, or 10% undertuning (out-of-phase mode corresponding to $\tilde{\omega}_2^{(n+1)}$) and in Fig. 3c for $\beta = +0.10$, or 10% overtuning (in-phase mode corresponding to $\tilde{\omega}_1^{(n+1)}$). One of the more interesting findings of this study is that there exists a small region of absorber undertuning values such that no system resonances occur. An example of this situation is shown in Fig. 3b for $\beta = -0.00351$. See [11] for a more thorough treatment of this feature and for a more detailed recommendation for the absorber tuning.

CONCLUDING REMARKS

An implementation of order-tuned vibration absorbers on a perfectly periodic bladed disk model has been investigated. A standard change of coordinates based on the cyclic symmetry of the system was employed to reduce the governing $2N$ equations of motion to a set of N , reduced-order equations, and an absorber tuning strategy was formulated based on these equations. It was shown that perfect absorber tuning—where the absorber tuning order is chosen to match the order of the excitation—completely eliminates the steady-state blade vibrations, and is effective at all rotational speeds of the rotor. It was also shown that the tuning can be made robust to parameter uncertainties by slightly undertuning the absorbers.

Generally, the blades on a turbomachinery rotor are meant to be identical. In practice, however, there are always small random uncertainties among the blades due to manufacturing tolerances, in-service wear, and so on. These small variations, or *mistuning*, can have a drastic effect on the forced response. In particular, they can lead to a confinement of vibration energy to a few blades or even a single blade, a phenomenon known as *localization* [12–14]. Due to this spacial confinement of energy, some of the blades may experience higher amplitudes than what is predicted from the ideal, perfectly periodic system [15–19].

The absorber tuning scheme presented here is based on an ideal system with identical blades and identical absorbers. The results offer a basic understanding of the effectiveness of order-tuned vibration absorbers on the forced response of a bladed disk, and serve as a first step to the investigation of more real-

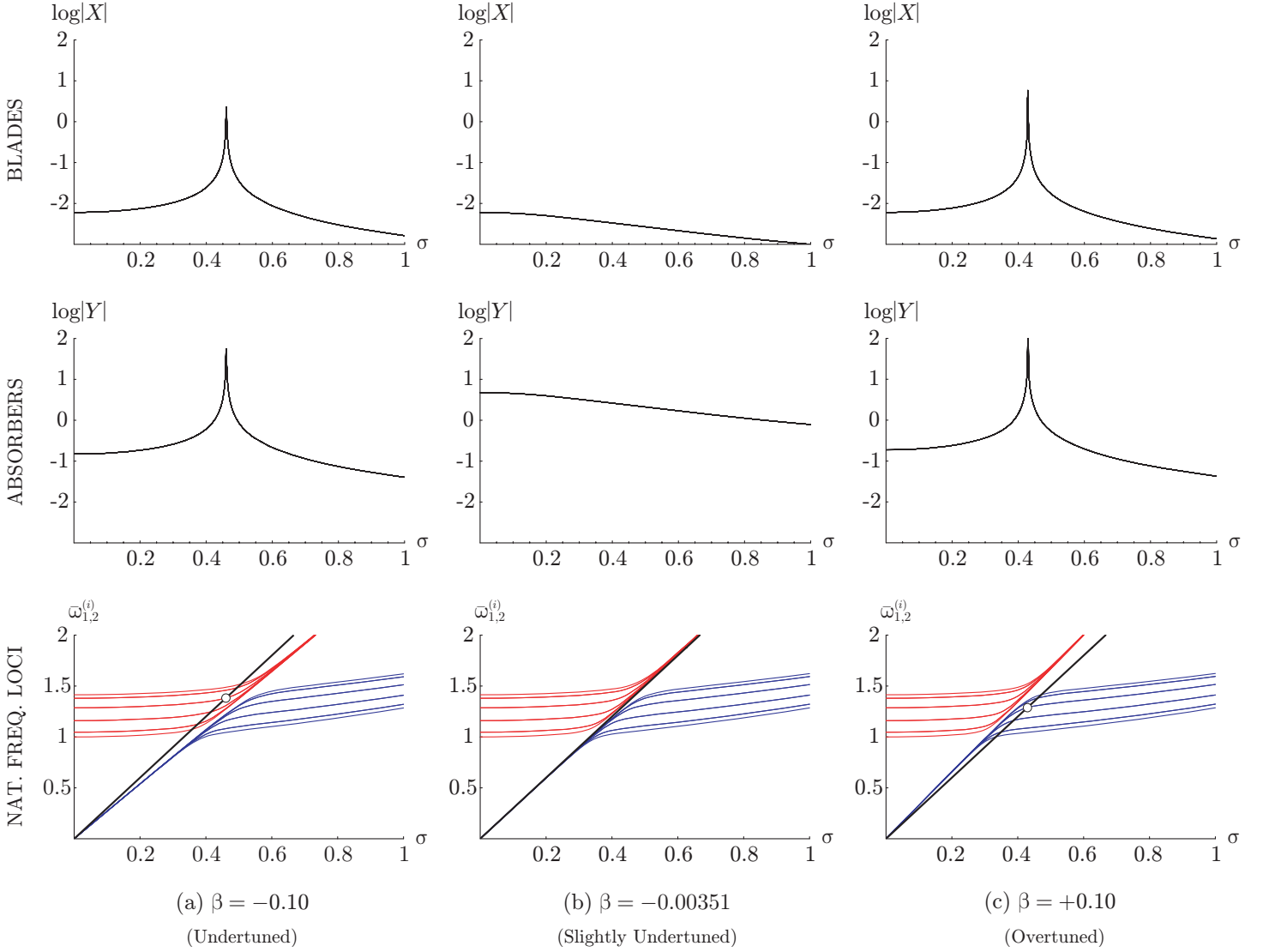


Figure 3. Absorber and blade frequency responses and Campbell diagrams for $N = 10$, $n = 3$, $\alpha = 0.84$, $\delta = 0.67$, $\mu = 0.015$, $\nu = 0.5$, $f = 0.01$, and $\xi_a = \xi_b = \xi_c = 0$: (a) -10% detuning ($\beta = -0.10$); (b) -0.351% detuning ($\beta = -0.00351$); (c) $+10\%$ detuning ($\beta = +0.10$). The logarithms are taken base 10.

istic models. Some topics for future work include general-path absorbers [20, 21], the effects of damping [10], and multi-DOF blade models [5] with, for example, nonlinear effects, imperfections [22], and intentional mistuning [23, 24].

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APPENDIX: MATHEMATICAL PRELIMINARIES

This appendix reviews some mathematical topics that may not be familiar to the reader, including the Kronecker product, the Fourier matrix, and the theory of circulant matrices. Special attention is given to the diagonalization of circulant and block circulant matrices, the results of which are used to decouple the governing equations of motion given by Eq. (7). A detailed account of the theory of circulants can be found in the work by Davis [6]. See the appendices of [8] for a more detailed summary of the relevant topics, including proofs to many of the ensuing results.

The Kronecker Product

Let the matrices \mathbf{A} and \mathbf{B} be $m \times n$ and $p \times q$, respectively. Then the Kronecker (direct) product of \mathbf{A} and \mathbf{B} is the $mp \times nq$ matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}. \quad (29)$$

If $m = n = p = q = 1$ (i.e., \mathbf{A} and \mathbf{B} are scalars) then $\mathbf{A} \otimes \mathbf{B}$ is simply scalar multiplication. If \mathbf{A} , \mathbf{B} , \mathbf{C} , and \mathbf{D} are square matrices, then

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}), \quad (30a)$$

$$(\mathbf{A} \otimes \mathbf{B})^* = \mathbf{A}^* \otimes \mathbf{B}^*, \quad (30b)$$

where $(\cdot)^*$ denotes the inverse $(\cdot)^{-1}$, transpose $(\cdot)^T$, or conjugate transpose $(\cdot)^H = (\bar{\cdot})^T$.

The Fourier Matrix

The $N \times N$ complex Fourier matrix is defined as

$$\mathbf{F}_N = [e_{ik}]; \quad e_{ik} = \frac{1}{\sqrt{N}} w^{(i-1)(k-1)}, \quad i, k = 1, \dots, N \quad (31)$$

where $w = e^{\frac{2j\pi}{N}}$ is the primitive N^{th} root of unity, and $j = \sqrt{-1}$. When the dimension of \mathbf{F}_N is clear, the subscript N will be omitted.

It is shown subsequently that all circulant matrices share the same linearly independent eigenvectors, the elements of which compose the N columns (or rows) of \mathbf{F} . They are denoted by the column vectors

$$\mathbf{e}_i = \frac{1}{\sqrt{N}} \left(1, w^{(i-1)}, w^{2(i-1)}, \dots, w^{(N-1)(i-1)} \right)^T, \quad i = 1, \dots, N. \quad (32)$$

An important property of the Fourier matrix is that it is unitary and, therefore,

$$\mathbf{F}^H \mathbf{F} = \mathbf{F} \mathbf{F}^H = \mathbf{I}, \quad (33)$$

where \mathbf{I} is the identity matrix. Finally, if \mathbf{F} is unitary, then so are the matrices $\mathbf{F} \otimes \mathbf{I}$ and $(\mathbf{F} \otimes \mathbf{I})^H = \mathbf{F}^H \otimes \mathbf{I}$.

Circulant Matrices

An $N \times N$ circulant matrix (or *circulant* for short) is formed from an N -vector by cyclically permuting its entries, and is of the form

$$\mathbf{C} = \begin{bmatrix} c_1 & c_2 & \cdots & c_N \\ c_N & c_1 & \cdots & c_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{bmatrix}. \quad (34)$$

Thus a circulant matrix is defined completely by an ordered set of *generating elements* c_1, c_2, \dots, c_N . It is convenient to define the *circulant operator* $\text{circ}(\cdot)$ that takes as its argument these generating elements and results in the array given by Eq. (34). That is,

$$\mathbf{C} = \text{circ}(c_1, c_2, \dots, c_N). \quad (35)$$

An $NM \times NM$ block circulant matrix is defined similarly to Eq. (34) and has the representation given by Eq. (35), where each entry c_k is replaced by the $M \times M$ matrix \mathbf{C}_k for $k = 1, \dots, N$.

Diagonalization of Circulants

Equation (34) can be diagonalized via the unitary (similarity) transformation

$$\mathbf{F}_N^H \mathbf{C} \mathbf{F}_N = \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \lambda_2 & \\ & & \ddots \\ \mathbf{0} & & & \lambda_N \end{bmatrix}, \quad (36)$$

where

$$\lambda_i = \sum_{k=1}^N c_k w^{(k-1)(i-1)}, \quad i = 1, \dots, N. \quad (37)$$

As a consequence, *all* circulant matrices share the same eigenvectors, which are given by Eq. (32). Their eigenvalues are given by Eq. (37) and depend on the generating elements c_1, c_2, \dots, c_N .

An $NM \times NM$ block circulant matrix \mathbf{C} with $M \times M$ generating matrices $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_N$ can be block diagonalized via the unitary (similarity) transformation

$$(\mathbf{F}_N^H \otimes \mathbf{I}_M) \mathbf{C} (\mathbf{F}_N \otimes \mathbf{I}_M) = \begin{bmatrix} \Lambda_1 & & \mathbf{0} \\ & \Lambda_2 & \\ & & \ddots \\ \mathbf{0} & & & \Lambda_N \end{bmatrix}, \quad (38)$$

where \mathbf{I}_M is the $M \times M$ identity matrix and

$$\Lambda_i = \sum_{k=1}^N \mathbf{C}_k w^{(k-1)(i-1)}, \quad i = 1, \dots, N. \quad (39)$$

Since Eq. (38) is a unitary transformation, it preserves the eigenvalues of \mathbf{C} . Hence its eigenvalues are the eigenvalues of the N , $M \times M$ matrices Λ_i .