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NONLINEAR NORMAL MODES OF A ROTATING SHAFT BASED ON THE INVARIANT MANIFOLD METHOD

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ABSTRACT

The nonlinear normal mode methodology is generalized to the study of a rotating shaft supported by two short journal bearings. For rotating shafts, the forces arising from the supporting hydraulic bearings are nonlinear, even when the shaft deformation is in the linear range. In this study, the rotating shaft is represented by a linear beam, while a simplified bearing model is employed so that the nonlinear supporting forces can be expressed analytically. The equations of motion of the coupled shaft-bearings system are constructed using the Craig-Bampton method of component mode synthesis, producing a model with as few as six degrees of freedom (DOF). Using an invariant manifold approach, the individual nonlinear normal modes of the shaft-bearings system are then constructed, yielding a single-DOF reduced-order model for each nonlinear mode. A generalized formulation for the manifolds is required, since the system features damping as well as gyroscopic and nonconservative circulatory terms. The nonlinear modes are calculated numerically using a nonlinear Galerkin method that is able to capture large amplitude motions. The shaft response from the nonlinear mode model is shown to match extremely well simulations from the reference Craig-Bampton model.

INTRODUCTION

Many rotating structures are supported by devices that are inherently nonlinear, e.g., journal bearings. The dynamic analysis of nonlinear rotating systems has been the subject of a number of studies, for example, those of Yamanchi [1] and Kim and Noah [2,3], who employed the method of harmonic balance. Choi and Noah [4] added discrete Fourier transform procedures to the harmonic balance method and also included subharmonic response components. Dynamic condensation techniques were used with the harmonic balance method by Kim and Noah [3] to reduce the size of the system models. The objective of the present work is to develop reduced-order models of nonlinear shaft-bearings systems using the invariant manifold-based nonlinear normal mode methodology.

Nonlinear normal modes (NNMs) provide a general framework for the construction of reduced-order models for nonlinear systems. The concept of NNMs was first introduced by Rosenberg [5] with the study of conservative, symmetric, nonlinear systems. A NNM was defined in the configuration space so that its application was strictly limited to systems without gyroscopic effects and damping. Shaw and Pierre [6] extended the definition of nonlinear normal modes using invariant manifold techniques, wherein a NNM is defined as a two-dimensional invariant surface in the phase space, which is tangent to the hyperplane that represents the corresponding mode of the linearized model. The NNM response is thus captured by a single-degree of freedom (DOF) nonlinear oscillator. Using this more general definition, a systematic construction method for NNMs has been proposed by Boivin *et al.* [7] and Pesheck *et al.* [8] for nonlinear systems with quadratic and cubic nonlinearities, including systems with a large number of DOFs and for large amplitude motions. Nayfeh and Nayfeh [9] also used the invariant manifold approach to construct NNMs using perturbation methods for weakly nonlinear systems.

In this paper, the NNMs are constructed for an established model of a nonlinear shaft-bearings system, which consists of a linear rotating shaft supported by short nonlinear journal bearings at its two ends. Since the linearized system is gyroscopic, damped, and (nonconservative) circulatory—thus featuring a non-symmetric stiffness matrix—, an extension of the invariant manifold approach is required in order to accommodate these effects. The NNMs are shown to provide very accurate reduced-order models of the shaft-bearings system.

The paper is organized as follows. In the first section, the mathematical model of the shaft-bearings system is derived using component mode synthesis (CMS). Then, in the second section, the vibration modes of the linearized system are investigated. Finally, in the third section, the theory of NNMs is extended to systems whose linearized counterpart is gyroscopic, nonconservative, and circulatory, and the individual NNM invariant manifolds of the nonlinear shaft-bearings system are calculated.

1. THE ROTATING SHAFT-BEARINGS SYSTEM

A diagram of the system of interest, a rotating shaft supported by short journal bearings at its two ends, is shown in Fig. 1. A Rayleigh beam with uniform cross-section properties is used to model the shaft, which is assumed to be perfectly balanced. The forces created by the oil film in the bearings are nonlinear and can be represented as nonlinear boundary conditions for the beam.

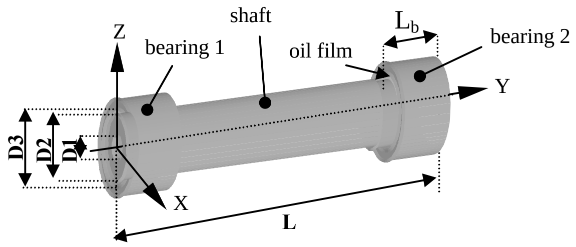

Figure 1: Schematic of the shaft-bearings system

In Fig. 1, the inertial frame, R_{XYZ}, is fixed in space and the Y axis passes through the centers of the bearings at the two ends. The shaft is defined by its length L = 1*m*, outer diameter D_2 = 0.0592*m*, and inner diameter D_1 = 0.02*m*. The nominal clearance between the bearing and the shaft is set as $c = (D_3 - D_2)/2 = 51.10^{-6}m$, where D_3 is the inner diameter for both bearings. The length of the bearing is L_b = 0.0285*m*, and the dynamic viscosity of the oil film is chosen as m = 0.0068 $N.s/m^2$.

1.1 THE SHAFT MODEL

From Rayleigh beam theory, the kinetic energy, T, and strain energy, U, of the rotating shaft are given by, respectively:

$$T = \frac{rA}{2}\int_0^L (\dot{u}^2 + \dot{w}^2)dy + \frac{rI}{2}\int_0^L \left(\left(\frac{\partial \dot{u}}{\partial y}\right)^2 + \left(\frac{\partial \dot{w}}{\partial y}\right)^2 \right) dy$$

$$+ rIL\Omega^2 - 2rI\Omega \int_0^L \frac{\partial \dot{u}}{\partial y}\frac{\partial w}{\partial y} dy \tag{1}$$

$$U = \frac{EI}{2}\int_0^L \left(\left(\frac{\partial^2 u}{\partial y^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \right) dy \tag{2}$$

where $u(y,t)$ and $w(y,t)$ are the displacements of the points on the neutral axis of the shaft in the X and Z directions, respectively, Ω is the constant angular velocity of the shaft, A is the shaft's cross-sectional area, and I is the second area moment of inertia of the cross section. The material parameters for the shaft are its Young's modulus $E = 2.1\ 10^{11} Pa$ and mass density r = 7800 kg/m^3.

In order to obtain an efficient discretized model for the rotating shaft, the Craig-Bampton method [10] of component mode synthesis is applied, where the shaft is the linear substructure constrained at its ends by the nonlinear bearings. This allows for the nonlinear effects of the supporting bearings to be captured solely by the DOFs corresponding to the constraint modes. The shaft displacements u and w are thus each expanded as a linear combination of the modes of free vibration of the shaft pinned at both ends and two constraint modes, each corresponding to a rigid body motion of the shaft induced by a unit displacement at one of its ends:

$$u(y,t) = \sum_{i=1}^{m} \Phi_i(y) a_i(t)\ ; \qquad w(y,t) = \sum_{i=1}^{m} \Phi_i(y) b_i(t) \tag{3}$$

where $\Phi_i(y) = \begin{cases} \sin\left(\frac{ip\ y}{L}\right) & i = 1,\dots,m-2 \\ y/L & i = m-1 \\ 1 - y/L & i = m \end{cases}$

where the first $m-2$ expansion functions are the mode shapes of the simply supported shaft and the final two are the static constraint modes. We substitute the expansion functions for u and w into Hamilton's principle, yielding:

$$\int_{t_1}^{t_2} (dT - dU + dW)dt = 0 \qquad \forall (t_1, t_2) \tag{4}$$

where dW is the virtual work done by the gravity force and the nonlinear supporting forces of the two bearings. The resulting discretized model is given as:

$$[M_1]\{\ddot{x}\} + [G_1]\{\dot{x}\} + [K_1]\{x\} = \{F\} \tag{5}$$

where

$$\{x\} = \begin{Bmatrix} \{a\} \\ \{b\} \end{Bmatrix},\quad [M_1] = \left(rS \begin{bmatrix} [A] & [0] \\ [0] & [A] \end{bmatrix} + rI \begin{bmatrix} [B] & [0] \\ [0] & [B] \end{bmatrix} \right)$$

$$[G_1] = 2rI\Omega \begin{bmatrix} [0] & [B] \\ -[B] & [0] \end{bmatrix},\ \text{and}\ [K_1] = EI \begin{bmatrix} [C] & [0] \\ [0] & [C] \end{bmatrix}$$

Here $\{a\}$ and $\{b\}$ are m-vectors of the generalized coordinates generated by the CMS formulation, a_i and b_i, respectively. The matrices $[A]$, $[B]$ and $[C]$ are listed as follows:

$$[A] = \int_0^L [\Phi(y)]^T [\Phi(y)] dy \qquad [B] = \int_0^L [\Phi_{,y}(y)]^T [\Phi_{,y}(y)] dy$$

$$[C] = \int_0^L [\Phi_{,yy}(y)]^T [\Phi_{,yy}(y)] dy$$

The matrices $[M_1]$, $[G_1]$ and $[K_1]$ are the inertia, gyroscopic, and stiffness matrices, respectively. Note that $\{x\}$ is defined with respect to the inertial frame R_{XYZ}. The force vector $\{F\}$ has the form:

$$\{F\} = -rgA\int_0^L \{\Phi_i(y)\} dy + \{F_2(x,\dot{x})\} = \{F_1\} + \{F_2(x,\dot{x})\} \tag{6}$$

where the first term is the gravitational load, and the second term $\{F_2\}$ is the supporting force at the journal bearing, which is considered next.

1.2 THE BEARING FORCE

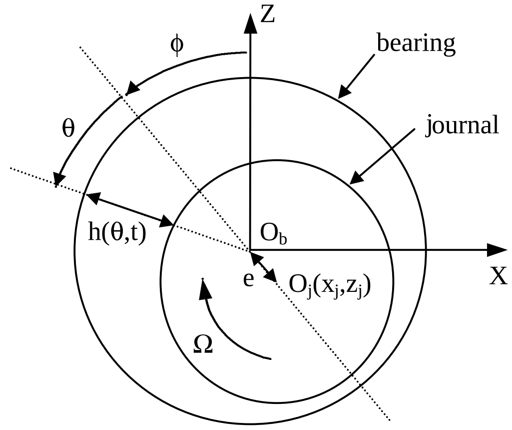

Figure 2: Schematic of the journal bearing model

In Fig. 2, R_{XYZ} is the inertial frame and Ω is the angular velocity, defined in Fig. 1. The nominal thickness of the oil film is h, e is the eccentricity between the bearing axis and the shaft axis, and f is the attitude angle of the line connecting the bearing and shaft centers with respect to the Z axis. The horizontal and vertical displacements of the center of the shaft (the journal) in the bearing are denoted as x_j and z_j, respectively.

The thickness of the oil film, h, can be expressed as:

$$h = c - z_j \cos(q + f) + x_j \sin(q + f) \tag{7}$$

where c is the nominal clearance between the shaft and the bearing. Based on Reynolds' equation, the pressure of the oil film can be modeled as:

$$\frac{\partial}{\partial y}\left(\frac{h^3}{6m}\frac{\partial p}{\partial y}\right) + \frac{1}{R^2}\frac{\partial}{\partial q}\left(\frac{h^3}{6m}\frac{\partial p}{\partial q}\right) = \Omega\frac{\partial h}{\partial q} + 2\frac{\partial h}{\partial t} \tag{8}$$

In Eq. (8), m is the fluid viscosity, R is the outer radius of the beam and p is the fluid film pressure. The short bearing assumption implies that R^2 is preponderant in Eq. (8), so that the second term on the left hand side in Reynolds' equation can be neglected. This yields

$$\frac{h^3}{6m}\frac{\partial^2 p}{\partial y^2} = \Omega(z_j \sin(q + f) + x_j \cos(q + f)) - 2\left(\dot{z}_j \cos(q + f) - \dot{x}_j \sin(q + f)\right) \tag{9}$$

The boundary conditions over y are $p(q,0) = p(q,L_b) = 0$, where L_b is the length of the bearing and q is integrated over $[0, p]$ instead of $[0, 2p]$ due to cavitation effects [11]. Then the two forces created by the pressure field are the integrals of the pressure over the fluid-film surface contact. A dimensionless analysis of the problem leads to the following definitions:

$$Z_j = \frac{z_j}{c}, \quad X_j = \frac{x_j}{c}, \quad \dot{Z}_j = \frac{\dot{z}_j}{\Omega c}, \quad \dot{X}_j = \frac{\dot{x}_j}{\Omega c}$$

$$H_j = 1 - Z_j \cos(q + f) + X_j \sin(q + f) \tag{10}$$

where H_j is the dimensionless fluid-film thickness.

Then, the resultant forces F_X and F_Z in the X and Z directions can be obtained as [12]:

$$F_x = -\frac{mRL^3\Omega}{2c^2}\int_0^p \left(\frac{Z_j \sin(q + f) + X_j \cos(q + f)}{H_j^3} - \frac{2(\dot{Z}_j \cos(q + f) - \dot{X}_j \sin(q + f))}{H_j^3}\right)\sin(q + f)\,dq \tag{11.a}$$

$$F_Z = \frac{mRL^3\Omega}{2c^2}\int_0^p \left(\frac{Z_j \sin(q + f) + X_j \cos(q + f)}{H_j^3} - \frac{2(\dot{Z}_j \cos(q + f) - \dot{X}_j \sin(q + f))}{H_j^3}\right)\cos(q + f)\,dq \tag{11.b}$$

Both forces are nonlinear in the displacements and velocities of the shaft's ends, involving complicated integrals that can be obtained analytically using commercial mathematical software such as Maple®.

2. THE LINEARIZED MODEL

Complex modal analysis is applied to the system linearized about its equilibrium position, to determine the natural modes of vibration and the shaft's first critical speed.

2.1 THE EQUILIBRIUM POSITION

The dimensionless equilibrium position of the shaft center is given by Ocvirck [13] as:

$$\begin{cases} Z_e = -e_e \cos f_e \\ X_e = e_e \sin f_e \end{cases} \quad \text{where} \quad f_e = \arctan\left(\frac{p\sqrt{1 - e_e^2}}{4e_e}\right) \tag{12}$$

and the dimensionless eccentricity $e_e = e_e / c$ is the solution of:

$$\frac{mN}{f}\frac{R}{2c^2}L_b^3 = \frac{(1 - e^2)^2}{pe\sqrt{p^2(1 - e^2) + 16e^2}} \tag{13}$$

where N (with $\Omega = 2\pi N$) is the angular frequency and f is the load (in this study it is given by the weight of the beam). By linearizing the bearing forces with respect to this equilibrium position, the stiffness matrix $[K_2]$ (which is not symmetric) and damping matrix $[D_2]$ can be obtained as:

$$K_{2IJ} = \left.\frac{\partial F_I}{\partial J}\right|_e \quad \text{and} \quad D_{2IJ} = \left.\frac{\partial F_I}{\partial \dot{J}}\right|_e \quad \text{with } I = (X,Z) \text{ and } J = (X,Z)$$

where $Z=Z_j-Z_e$ and $X=X_j-X_e$. The eight coefficients for these matrices are given in [12].

2.2 THE LINEAR MODES

In order to study the eigenvectors of this system, Eq. (5) is linearized with respect to the equilibrium position. The linear system can be written in the general following form:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\} \tag{14}$$

where $[M] = [M_1]$, $[C] = [G_1] - [D_2]$, $[K] = [K_1] - [K_2]$, and $\{x\}$ is defined relative to the equilibrium position.

Complex modal analysis can be applied to the state space formulation of the system. Defining the velocity vector as $\{y\}$, Eq. (14) can be written in first-order form as:

$$\{\dot{z}\} = [D]\{z\} \tag{15}$$

where $\{z\} = \begin{Bmatrix} \{x\} \\ \{y\} \end{Bmatrix}$ and $[D] = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix}$.

Since the matrix $[D]$ is real, those eigensolutions that are complex must occur in conjugate pairs.

Since the NNMs are to be defined in terms of real modal coordinates, a traditional complex mode transformation matrix would lead to difficulties in constructing the corresponding invariant manifolds. Thus, herein a real modal transformation is chosen that renders the matrix $[D]$ block diagonal [14]. This is achieved by constructing the matrix $[Z']$, whose columns are the real eigenvectors and the real and the imaginary parts of the complex eigenvectors, as follows:

$$[Z'] = \left[\Re\bar{z}_1 \quad \Im\bar{z}_1 \quad \ldots \quad \Re\bar{z}_p \quad \Im\bar{z}_p \quad \bar{z}_{2p+1} \quad \bar{z}_{2N}\right] \tag{16}$$

where N is the total number of DOF for the rotating shaft system, Eq. (5), and is equal to $2m$.

Letting $\{z(t)\} = [Z']\{h(t)\}$, Eq. (15) is transformed into $\{\dot{h}(t)\} = [\Lambda']\{h(t)\}$, with

$$[\Lambda'] = \begin{bmatrix} \Re l_1 & \Im l_1 & & & & & \\ -\Im l_1 & \Re l_1 & & & & 0 & \\ & & \ddots & & & & \\ & & & \Re l_p & \Im l_p & & \\ & & & -\Im l_p & \Re l_p & & \\ & & & & & l_{2p+1} & \\ & 0 & & & & & \ddots \\ & & & & & & l_{2N} \end{bmatrix}$$

The resulting first-order equations are at worst pairwise coupled. Those equations that are uncoupled are associated with the real eigenvalues and correspond to overdamped modes, whose dynamics occur in a one-dimensional linear subspace. Motions in these modes are decaying and non-oscillatory. The modes associated with the complex conjugate pairs of eigenvalues correspond to the 2×2 diagonal blocks of $[\Lambda']$ and are for underdamped, oscillatory modes. The dynamics of these modes occur in two-dimensional subspaces. Motions in the underdamped modes consist of decaying oscillations, which are described by pairs of first-order differential equations:

$$\begin{cases} \dot{s}_i = -z_i w_i s_i + w_i\sqrt{1-z_i^2}\,t_i \\ \dot{t}_i = -w_i\sqrt{1-z_i^2}\,s_i - z_i w_i t_i \end{cases} \qquad i = 1,\ldots,p \tag{17}$$

where w_i is the undamped natural frequency of the mode and z_i its modal damping ratio, both of which are defined from $\Re l_i = -z_i w_i$ and $\Im l_i = w_i\sqrt{1-z_i^2}$. $s_i = h_{2i-1}$ and $t_i = h_{2i}$ are the components of $\{h\}$ associated with the ith underdamped mode.

2.3 STABILITY OF THE LINEAR MODEL

In the Lyapunov sense, the system is unstable if the real part of any eigenvalue is positive. The linear motion associated with such an instability becomes infinitely large with time and thus linear theory breaks down. The eight coefficients of the linear model given in reference [12] are strongly linked to the stability of the system, especially the cross-coupled stiffness terms. In this study we have examined the stability of the shaft-bearings system using two, four, and six normal modes of vibration to describe the shaft motion, corresponding respectively to shaft-bearings models with six, eight and ten DOFs. The results for these three models have been found to be in close agreement, indicating that a single normal mode in each direction is sufficient to represent the shaft's motion. This makes sense since the shaft is balanced and thus the first beam mode is dominant. Therefore, the six-DOF model is employed in the remainder of the paper.

Figure 3 shows that the linearized system is unstable when the angular velocity of the shaft is larger than 375 *rad/s*.

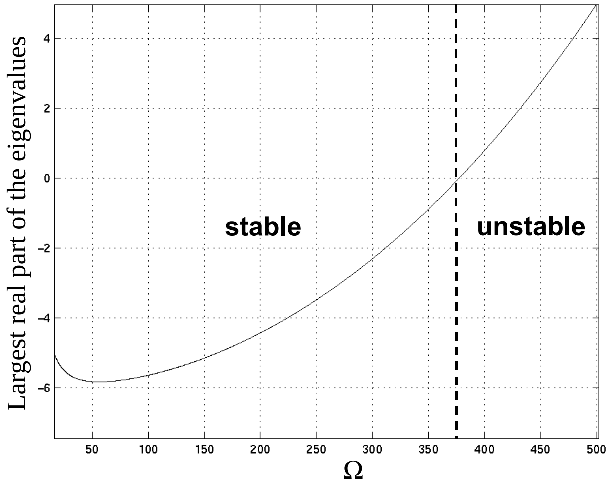

Figure 3: Stability of the linear model: largest real part of the eigenvalues versus Ω

2.4 LINEAR MODE MOTIONS

As the convergence study indicated, a six-DOF model is sufficient to capture accurately the lower oscillating modes of the system. Recall that this model includes one free vibration normal mode and two static constraint modes for the shaft in each of the X and Z directions. The eigenanalysis was carried out in the state space in terms of the shaft angular velocity. It was found that up to approximately Ω = 550 *rad/s*, the shaft-bearings system possesses four one-dimensional overdamped modes, corresponding to real eigenvalues, and four two-dimensional oscillatory modes, corresponding to complex eigenvalues. Two of the four overdamped modes become a single two-dimensional oscillatory mode as Ω grows beyond 550 *rad/s*, and the last two overdamped modes turn into a sixth oscillatory mode for Ω > 700 *rad/s*. Since the complex modes correspond to oscillatory motions of the system, we focus on them in the following, at the rotation speed Ω = 300 *rad/s*..

The physical nature of the modes can be depicted as follows. In Fig. 4-7, curves are drawn for each oscillating mode in the X and Z coordinates that display the displacements of both ends of the beam as they move in the bearings during a modal motion. The accompanying sketches of the shaft show it as straight, whereas in reality it deflects in its first vibration mode (although its amplitude is relatively small).

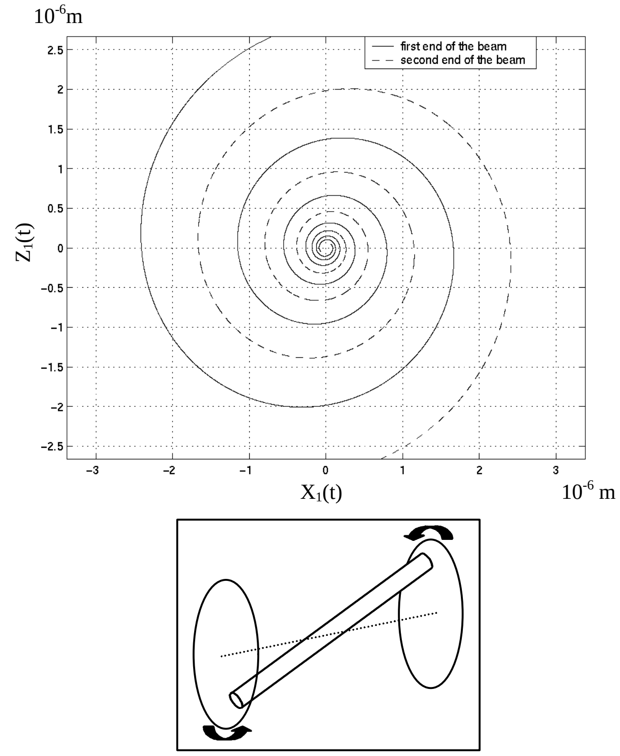

Figure 4: First mode shape
ω_1 = 50.647, ζ_1 =0.1166

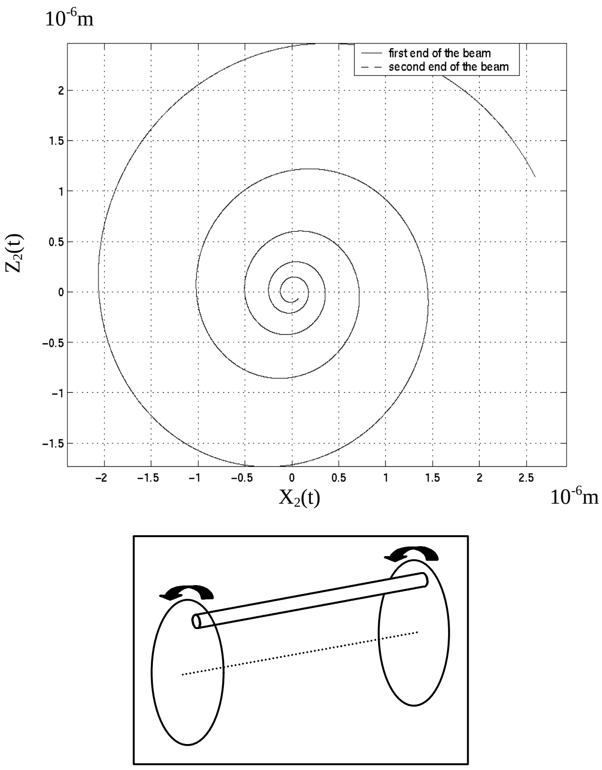

Figure 5: Second mode shape
ω_2 = 50.676, ζ_2 = 0.1112

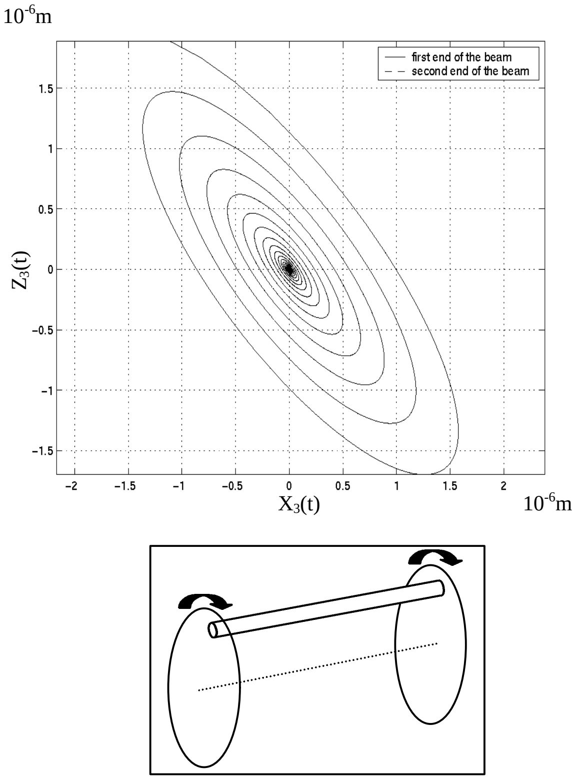

Figure 6: Third mode shape
w_3 =775.495, ζ_3= 0.0457

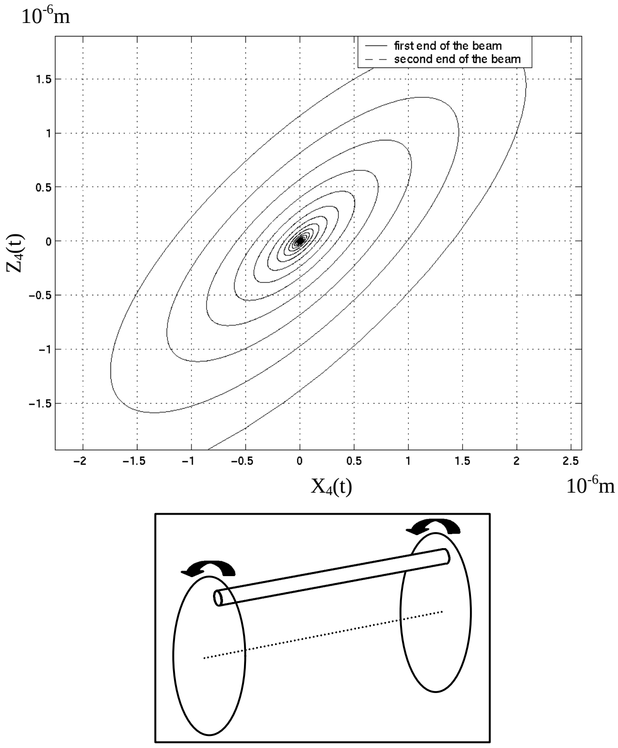

Figure 7: Fourth mode shape
ω_4 = 775.684, ζ_4 = 0.0561

2.5 COMPARISON BETWEEN NONLINEAR AND LINEAR MODELS

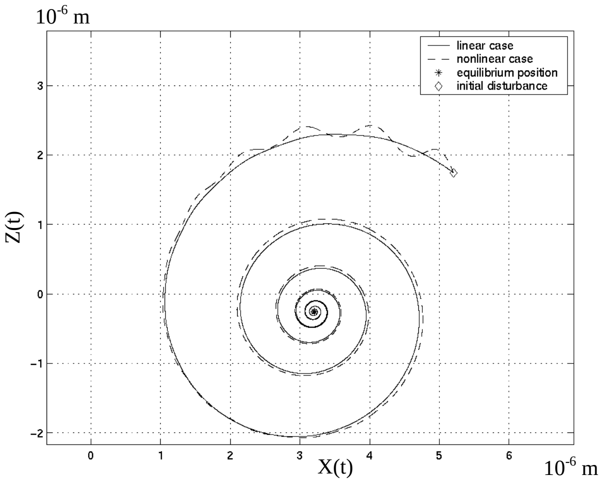

Figure 8: Trajectory of one end of the shaft for a very small initial disturbance and Ω = 100 *rad/s*.

Using a very small disturbance from the equilibrium position as initial conditions for both the linear and nonlinear models, the time simulation results are seen to be very close in Fig. 8. For Ω = 100 *rad/s*, which lies in the stable range, after a short time the beam settles to its equilibrium position.

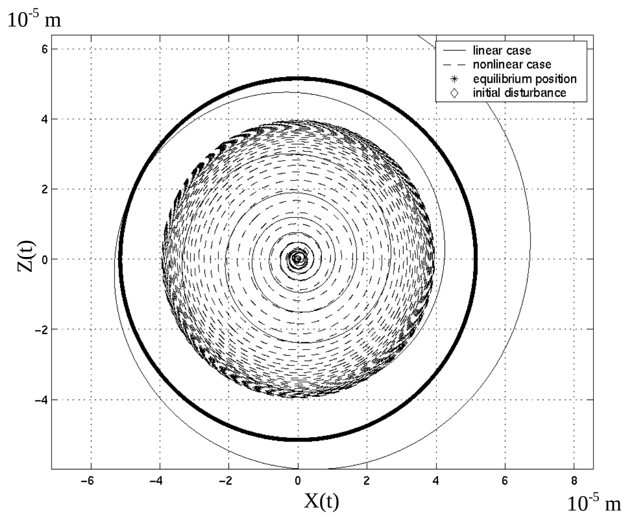

Figure 9: Trajectory of one end of the shaft for a very small initial disturbance and Ω = 2550 *rad/s*.

As seen in Fig. 9, for a much larger shaft rotation speed, Ω = 2550 *rad/s*, the linear model is unstable, as the angular velocity is then greater than the critical speed, $\Omega_c \approx 375 rad/s$, whereas the nonlinear model reaches a limit cycle corresponding to oil whirl [15]. For a lightly loaded shaft, the whirl occurs at a frequency that is half the shaft's angular velocity. The thick black line in Fig. 9 represents the largest displacement of the center point of shaft at the bearing location that is kinematically allowed. As shown in Fig. 9, the oil whirl phenomenon cannot be predicted by linearized stability theory.

3. NONLINEAR NORMAL MODES

For a linear system, an important property of a modal motion is that if one knows the motion of a single generalized coordinate, then the motions of all others coordinates are specified by that mode's eigenvector. The same property is used for the nonlinear case, wherein a motion in a nonlinear normal mode is defined such that all displacements and velocities are functionally related to a single displacement-velocity pair. In a NNM the nonlinear system thus behaves essentially as a single-DOF oscillator. Even if NNMs are close to the usual linear normal mode concept, similarities stop there. Many results given by NNMs have no counterparts in linear systems, such as internal resonances [16], where two NNMs exchange energy during the motion.

Following the concept of invariant manifolds, Shaw and Pierre [6] defined a NNM as a motion that lies on a two-dimensional invariant manifold in the system's phase space. Here "invariant" indicates that any motion initiated on the manifold will remain on it all the time. A single displacement-velocity pair is chosen as master coordinates, which characterize the individual nonlinear mode motion. All the remaining "slave" coordinates are parameterized by the two master coordinates and give the constrained conditions. Previous work on the invariant manifold method has been restricted to systems with diagonal stiffness and damping matrices in the modal coordinates space.

In order to construct NNMs for the rotating shaft system, a new formulation has been introduced for obtaining the PDEs governing the invariant manifold. This new formulation is more general and can be applied to systems with non-proportional damping forces, gyroscopic effects, and non-symmetric circulatory stiffness matrices.

3.1 NNM FORMULATION

In this part we use Eq. (5) written with respect to the equilibrium position. This equation becomes:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}+[K_1]\{x_e\}=\{F_{NL}(x+x_e,\dot{x})\}$$

and

$$\{F_{NL}(x+x_e,\dot{x})\}=\{F_2(x+x_e,\dot{x})\}-[K_2]\{x\}-[D_2]\{\dot{x}\}+\{F_1\}$$

All the matrices and forces in this equation have been defined in Sections 2.1 and 2.2, and $\{x_e\}$ is the equilibrium position. Then $\{x\}$ is defined with respect to the equilibrium position.

The new set of master coordinates is chosen as:

$$\begin{cases} s_k = a\cos f \\ t_k = a\sin f \end{cases} \tag{18}$$

where s_k and t_k are one pair of real linear coordinates defined in Eqs. (16) and (17), chosen as one of the oscillating modes. It is assumed that the real linear transformation can be written as follows:

$$[\Lambda']=\begin{bmatrix} a_1 & b_1 & & & \\ b_2 & a_2 & & & \\ & & \ddots & & \\ & & & a_{2N-1} & b_{2N-1} \\ & & & b_{2N} & a_{2N} \end{bmatrix} \tag{19}$$

$$\begin{cases} a_k = a_{k+1} \\ b_k = -b_{k+1} \end{cases} k = 1, \dots 2p-1 \qquad \begin{cases} a_k \neq a_{k+1} \\ b_k = b_{k+1} = 0 \end{cases} k = 2p, \dots, 2N-1$$

Taking in account the vector of nonlinear forces and the real linear transformation, the equations of coherence are:

$$\begin{cases} \dot{s}_k = \dot{s}_k \Leftrightarrow \dot{a}\cos f - a\dot{f}\sin f = a_{2k-1}a\cos f + b_{2k-1}a\sin f + f_{2k-1} \\ \dot{t}_k = \dot{t}_k \Leftrightarrow \dot{a}\sin f + a\dot{f}\cos f = b_{2k}a\cos f + a_{2k}a\sin f + f_{2k} \end{cases} \quad (20)$$

with nonlinear forces:

$$\{f\} = [Z']^{-1} \begin{Bmatrix} \{0\} \\ \{F_{NL}(\{x(h,\dot{h})\}, \{\dot{x}(h,\dot{h})\})\} \end{Bmatrix}$$

Equation (20) yields:

$$\begin{cases} \dot{a} = a_{2k-1}a\cos^2 f + a_{2k}a\sin^2 f + f_{2k-1}\cos f + f_{2k}\sin f \\ \dot{f} = -b_{2k-1} + (a_{2k} - a_{2k-1})\sin f\cos f - \dfrac{f_{2k-1}\sin f}{a} + \dfrac{f_{2k}\cos f}{a} \end{cases} \quad (21)$$

Following the definition of the invariant manifold, all the other DOFs are constrained in the (a, j) coordinates:

$$\begin{cases} s_i = P_i(a,f) \\ t_i = Q_i(a,f) \end{cases} \quad i = 1, \dots, N\, i \neq k \quad (22)$$

The time-independent partial differential equations in P_i and Q_i governing the invariant manifold geometry can be written by combining Eq. (22) to the equations of motion:

$$\begin{cases} \dot{s}_i = \dfrac{\partial P_i}{\partial a}\dot{a} + \dfrac{\partial P_i}{\partial f}\dot{f} = a_{2i-1}s_i + b_{2i-1}t_i + f_{2i-1} \\ \qquad\qquad\qquad\qquad i = 1, \dots N \quad i \neq k \\ \dot{t}_i = \dfrac{\partial Q_i}{\partial a}\dot{a} + \dfrac{\partial Q_i}{\partial f}\dot{f} = b_{2i}s_i + a_{2i}t_i + f_{2i} \end{cases} \quad (23)$$

In those equations, $\dot{a}$ and $\dot{f}$ have to be replaced by expressions (21). P_i and Q_i are periodic in f due to the transformed coordinates.

A computation is necessary to approximate the manifold. As P_i and Q_i must be periodic in f due to the transformed coordinates, they are expanded as a double series in the amplitude and the phase:

$$\begin{cases} P_i(a,f) = \sum_{l=1}^{N_a}\sum_{m=1}^{N_f} C_i^{l,m} T_{l,m}(a,f) \\ \qquad\qquad\qquad\qquad i = 1, \dots, N \quad i \neq k \\ Q_i(a,f) = \sum_{l=1}^{N_a}\sum_{m=1}^{N_f} D_i^{l,m} U_{l,m}(a,f) \end{cases} \quad (24)$$

where the Cs and Ds are the unknown expansion coefficients and the $T_{l,m}$ and $U_{l,m}$ are known shape functions, typically products of functions of a over $[0, a_{\max}]$ and f over $[0, 2p]$. N_a and N_f are the number of expansion functions used in the a and f directions, respectively. The higher these numbers are, the more accurate the approximation is. Expansions (24) are substituted into Eq. (23) and a Galerkin projection (multiplication by test function and integration over the chosen domain i.e. $[0, a_{\max}]\mathrm{x}[0, 2p]$) is carried out. This yields a set of nonlinear algebraic equations in the Cs and Ds which, when solved, produce a solution that minimizes the error for the given expansion in a least squares sense.

3.2 RESULTS AND DISCUSSION

The above single-NNM formulation cannot handle systems with internal resonances, for which a multi-nonlinear manifold formulation is needed. Thus, it is necessary to know under what conditions an internal resonance can occur for the system.

An internal resonance can occur if there exists positive or negative integers m_1, m_2, m_3, m_4, …, m_n such that $m_1 w_1 + m_2 w_2 + \cdots + m_n w_n = 0$ [16]. In this case strong non-linear interactions between the corresponding modes may take place. To check for the presence of internal resonances in the shaft-bearings system, the loci of the eigenvalues of the linearized system are plotted in the complex plane, with the angular speed Ω as a parameter, as shown in Fig. 10. Here Ω varies from 0 to 4,000 *rad/s*, and four distinct branches of complex conjugate pairs of oscillatory eigenvalues are seen (the other modes fall outside the scale of the plot). In order to avoid a one-to-one internal resonance, the eigenvalues must not be close to one another. Note that such a resonance precisely takes place between mode 1 and 2 for Ω approximately between 0 and 200 *rad/s*. It was found that in this region, the single-mode manifolds could not be calculated, and a two-mode manifold would be needed to capture the internal resonance. Hence this region was avoided.

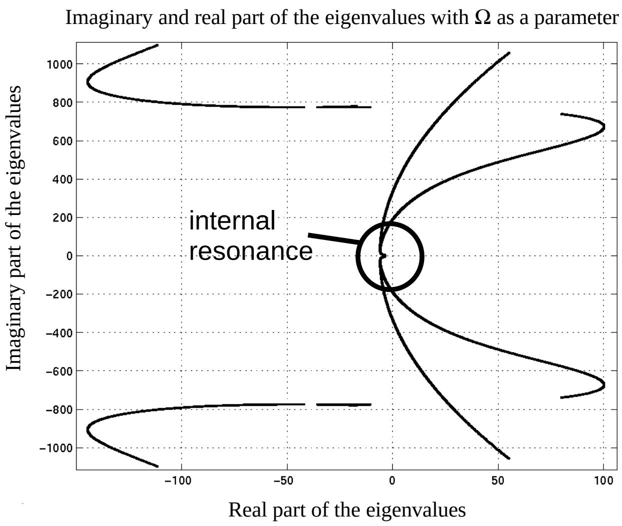

Figure 10: Internal resonance between mode 1 and mode 2

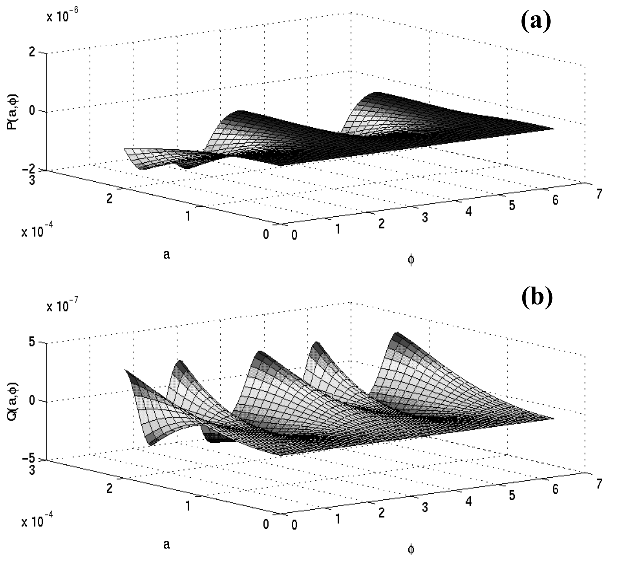

Figure 11: Invariant manifold of the two first components of the 4th nonlinear mode

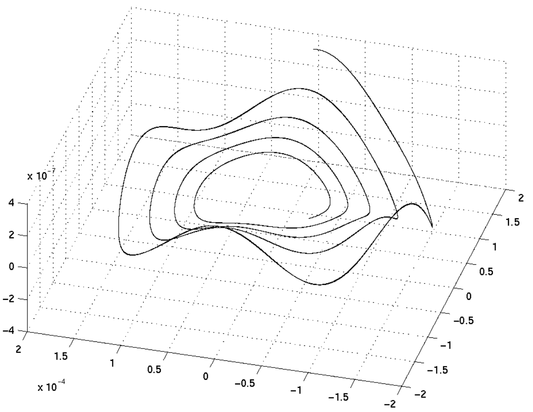

Figure 12: Exact and approximate solutions for t_1 versus s_4 and t_4

In relation to the above internal resonance, nonlinear manifolds were calculated for $\Omega = 300$ *rad/s*. Figure 11 shows the contribution of the first linear mode to the fourth nonlinear oscillating normal mode. Differences with the exact solution are undistinguishable, as illustrated in Fig. 12. Note that the latter is another possible view of a manifold, directly in the (s,t)–space instead of the (a,f)–space.

CONCLUSIONS

The invariant manifold approach has been extended successfully to describe the vibration behavior of nonlinear mechanical systems with general damping, gyroscopic and stiffness matrices. This new formulation uses a real linear transformation and a new set of coordinates in amplitude and phase. The collocation method used to solve for the invariant manifold geometry gives very good results and is quite fast. The methodology has been applied successfully to the construction of the NNMs of a rotating shaft with nonlinear bearings.

Although similar results could be obtained using classical methods for nonlinear systems (e.g., harmonic balance, multiple scales), the present approach produces systematically very accurate and minimal NNM reduced-order models. It holds significant promise for the study of realistic shaft models with mass imbalance, physical phenomena such as oil whirl, and internal resonances using multi-mode manifolds.

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