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# NON-LINEAR MODAL ANALYSIS OF STRUCTURAL SYSTEMS FEATURING INTERNAL RESONANCES

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## 1. INTRODUCTION

The superposition theorem is the cornerstone of linear system theory, which enables the study of the dynamic behavior of linear systems, both analytically and numerically. It is precisely this theorem which, by allowing modal analysis, capitalizes on the concept of normal modes of motions and renders them so important. By contrast, it is also precisely such a manner of recombining general motions from individual modal co-ordinates that is missing for non-linear dynamic systems (in particular, structural systems, on which this line of work is focused). Consequently, although the concept of non-linear normal modes of vibrations is well established for general, vibratory, non-linear, structural systems (see references [1–7]), their use in structural dynamics is restricted to very particular motions; namely, those involving a single mode response. Indeed, as superposition of individual modal responses cannot be performed, general motions of a non-linear structural system cannot usually be easily determined (except possibly for systems with similar non-linear modes). Such motions are typically handled by projecting the non-linear system on to a basis consisting of the modes of the linearized system.

However, it is possible to extend the definition of non-linear normal modes in a manner that enables a *non-linear modal analysis* of the free response of non-linear systems. By encompassing all modes of interest in a single multi-mode invariant manifold in the phase space—rather than separate single mode manifolds—one can create the framework needed to perform a reduced order modal analysis of non-linear systems, where interactions between the modelled modes are accounted for, while contaminations of (and from) the non-modelled modes is prevented—much like linear modal analysis of linear systems. This methodology is also perfectly suitable for dynamic (structural) systems with internal resonances. In this case, prior knowledge of any potential or existing internal resonances is not even required, as the procedure will exhibit them automatically. The number of equations to be simulated is the same as that of modelled modes in the multi-mode manifold. This non-linear modal analysis should be compared to performing a linear modal analysis of the non-linear system—a commonly employed technique—which often results in simulating a large number of equations to capture the various interactions between the linear modal co-ordinates. Also, the linear modal analysis of non-linear systems may completely overlook the occurrence of internal resonances—if internally resonant modes are not modelled simultaneously, numerical simulations can still be performed with potentially qualitatively inexact results—while the anomaly would be signalled by the present non-linear modal analysis procedure.

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In what follows, in section 2, this fundamentally new non-linear modal analysis technique is presented, and in section 3 its application to non-linear systems with internal resonances is described. Section 4 closes with a few conclusions.

## 2. MULTI-MODE INVARIANT MANIFOLDS

The modal analysis of linear dynamic systems possesses certain properties which, if generalizable to non-linear systems, would significantly facilitate their analysis. Some of the desirable features of a *non-linear modal analysis* are completeness (if all modes are included, the original system should be recovered), invariance of the modelled modes and non-contamination of (and from) the non-modelled ones (the non-modelled modes should remain unexcited and, thus, should not have to be simulated), reconstitution of general motions from modal co-ordinates, and recovery of single non-linear mode motions when a single mode is modelled.

Although completely different in spirit from the modal analysis of linear systems, the method described herein allows one to achieve the aforementioned properties. This non-linear modal analysis is defined geometrically in the system's phase space, in terms of a multi-dimensional invariant manifold which captures the dynamics of the various modeled modes (including the interactions between them), while it leaves the non-modelled modes quiescent for all time. Hence, the dynamics on the invariant manifold are entirely described by those of the modelled modes, and the solution obtained by simulation of only these modes is certain to be complete. Also, in the two limiting cases in which all modes are modelled or in which only one mode is modelled, the whole original system or a single non-linear normal mode, respectively, is recovered. As another interesting limiting case, the "traditional" modal analysis is obtained when this methodology is applied to linear systems, albeit in a detoured way.

The concept of non-linear normal modes of motions for non-linear structural systems has been defined geometrically, by taking advantage of the fact that such motions should inherently be (1) invariant and (2) described by two independent variables only (at least under some non-degeneracy conditions), the remaining variables being functionally related to the chosen pair (this functional relation is of course linear in the case of linear systems, while it is in general non-linear; see references [5, 6]). In the phase space, a non-linear normal mode is therefore described by a two-dimensional surface (planes in the case of linear systems). The dynamics of the system in such a mode are those of a single (second order) non-linear modal oscillator, and the invariance property ensures that a motion initiated in one non-linear mode remains in that mode for all time, and thus only that mode (i.e., only one second order modal oscillator) needs to be simulated.

While this approach is very well suited for single-mode approximations of a non-linear system, it is insufficient for more general motions. The method has therefore been generalized to higher-dimensional multi-mode invariant manifolds, where the influence of several modes can be accounted for. During general motions, the various modes included in the model will be coupled (as opposed to motions in a single non-linear normal mode, by definition), and these interactions will precisely be captured by this multi-mode manifold approach (even in the case of internal resonances; see section 3). On the other hand, the non-modelled modes will not be contaminated (due to the invariance property) and, thus, only the modelled modes need to be simulated.

It should be emphasized that the methodology presented hereafter was developed for discrete systems or discretized continuous systems, but its principle is generalizable to non-discretized continuous systems as well (although this might involve solving non-trivial multi-dimensional boundary value problems, which would probably still have to be discretized in practice). Notice also that the procedure directly attacks the ordinary differential equations of the discretized system without any further reference to their origin, so that the present non-linear modal analysis applies equally to mono-, bi- or tri-dimensional structural systems. In the following, the non-linear system will therefore be considered to be constituted of $N$ second order non-linear equations (or equivalently, of $2N$ first order equations). In the case of continuous systems, it is assumed for simplicity that the discretization was performed using the modes of the linearized system (about an operating point of interest), using a truncated Galerkin expansion procedure, so that the system is composed of a finite number of equations uncoupled at the linear order. Generically, the equations characterizing the system are assumed to be of the form

$$\dot{x}_i = y_i, \quad \dot{y}_i = f_i(x_1, \ldots, x_N, y_1, \ldots, y_N), \qquad \text{for } i = 1, \ldots, N. \tag{1}$$

When $M$ modes are to be modelled, a $2M$-dimensional manifold is defined (in the phase space) and, for most practical uses in structural mechanics (e.g., for weakly non-linear systems), a local approximation of it is constructed about the operating point of interest (this approximation will be tangent to the span of the $M$ linear modes, just as the actual invariant manifold is). The $2M$ modal variables required to describe the invariant manifold are chosen to be those corresponding to the $M$ linear modes to which the manifold has to be tangent, as

$$u_k = x_k, \quad v_k = y_k, \qquad \text{for } k \in S_m, \tag{2}$$

where $S_m$ denotes the subset of indices corresponding to the modelled modes. The $2N - 2M$ remaining variables are then functionally related to the modelled modes as

$$x_j = X_j(\mathbf{u}_m, \mathbf{v}_m), \quad y_j = Y_j(\mathbf{u}_m, \mathbf{v}_m), \qquad \text{for } j \notin S_m, \tag{3}$$

where $\mathbf{u}_m$ and $\mathbf{v}_m$ represent the vectors of the non-linear modal co-ordinates and velocities (i.e., they are the collections of the $u_k$'s and $v_k$'s, $k \in S_m$). As the original system of equations was cast into the linear modal co-ordinates, this procedure applied to a linear system yields $X_j(\mathbf{u}_m, \mathbf{v}_m) = Y_j(\mathbf{u}_m, \mathbf{v}_m) = 0$ for $j \notin S_m$ (independent linear modal co-ordinates), and the linear modal analysis is recovered. If the original system is non-linear, the Taylor series expansion of $X_j$ and $Y_j$ for $j \notin S_m$ has a vanishing linear part, reflecting the tangency property.

The constraint equations characterizing the geometry of the invariant manifold are then obtained by substitution of equation (3) into the $2N - 2M$ equations of motion corresponding to the non-modelled modes (equation (1) for $j \notin S_m$). In order to proceed, the time derivatives of $X_j$ and $Y_j$ need to be expanded using the chain rule, as

$$\dot{X}_j = \sum_{k \in S_m} \left[ \frac{\partial X_j}{\partial u_k} v_k + \frac{\partial X_j}{\partial v_k} f_k \right], \qquad \dot{Y}_j = \sum_{k \in S_m} \left[ \frac{\partial Y_j}{\partial u_k} v_k + \frac{\partial Y_j}{\partial v_k} f_k \right] \tag{4}$$

(since $\dot{u}_k = v_k$ and $\dot{v}_k = f_k$). The constraint equations are then given by

$$\left.\begin{aligned} \sum_{k \in S_m} \left[ \frac{\partial X_j}{\partial u_k} v_k + \frac{\partial X_j}{\partial v_k} f_k \right] &= Y_j \\ \sum_{k \in S_m} \left[ \frac{\partial Y_j}{\partial u_k} v_k + \frac{\partial Y_j}{\partial v_k} f_k \right] &= f_j \end{aligned}\right\}, \qquad \text{for } j \notin S_m, \tag{5}$$

where any occurrence of $x_l$ or $y_l$ (for $l \notin S_m$) in $f_k$ or $f_j$ is systematically replaced by $X_l$ or $Y_l$, respectively. For weakly non-linear systems, local approximations of the manifold about the operating point are found by means of Taylor series expansions of $X_j$ and $Y_j$ for $j \notin S_m$, as

$$
\begin{aligned}
X_j(\mathbf{u}_m, \mathbf{v}_m) = {} & \sum_{k \in S_m} [a^k_{1,j} u_k + a^k_{2,j} v_k] + \sum_{k \in S_m} \sum_{l \in S_m} [a^{k,l}_{3,j} u_k u_l + a^{k,l}_{4,j} u_k v_l + a^{k,l}_{5,j} v_k v_l] \\
& + \sum_{k \in S_m} \sum_{l \in S_m} \sum_{q \in S_m} [a^{k,l,q}_{6,j} u_k u_l u_q + a^{k,l,q}_{7,j} u_k u_l v_q + a^{k,l,q}_{8,j} u_k v_l v_q + a^{k,l,q}_{9,j} v_k v_l v_q] + \cdots, \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
Y_j(\mathbf{u}_m, \mathbf{v}_m) = {} & \sum_{k \in S_m} [b^k_{1,j} u_k + b^k_{2,j} v_k] + \sum_{k \in S_m} \sum_{l \in S_m} [b^{k,l}_{3,j} u_k u_l + b^{k,l}_{4,j} u_k v_l + b^{k,l}_{5,j} v_k v_l] \\
& + \sum_{k \in S_m} \sum_{l \in S_m} \sum_{q \in S_m} [b^{k,l,q}_{6,j} u_k u_l u_q + b^{k,l,q}_{7,j} u_k u_l v_q + b^{k,l,q}_{8,j} u_k v_l v_q + b^{k,l,q}_{9,j} v_k v_l v_q] + \cdots \qquad (7)
\end{aligned}
$$

(where, in fact, the linear part vanishes when the modes of the linearized system are used for the discretization). Like terms of the various modal co-ordinates and velocities are then gathered in equation (5), providing linear algebraic equations from which the Taylor series coefficients of $X_j$ and $Y_j$ are determined. These coefficients typically represent the influence of the $j$th linear mode on the multi-mode manifold defined by $S_m$, local to the operating point of interest (when only one mode is modelled, say $S_m = \{k\}$, they correspond to the influence of the $j$th linear mode on the $k$th non-linear mode).

Once the geometry of the multi-mode manifold is known (to some order), the dynamics of the system on it are obtained by considering the reduced set of equations of motion (equation (1)) corresponding to the $M$ modelled modes, i.e.,

$$\dot{u}_k = v_k, \quad \dot{v}_k = f_k(\mathbf{u}_m, \mathbf{v}_m), \qquad \text{for } k \in S_m, \qquad (8)$$

where a systematic substitution of $x_j$ and $y_j$ using equations (6) and (7) is performed for $j \notin S_m$. These equations are in general those of coupled second order non-linear oscillators, which accounts for potential modal interactions. However, they are uncoupled (at least to the order at which one is working) from the non-modelled modes which will therefore not be contaminated and need not be simulated. The dynamics of the $2N - 2M$ components, which do not correspond to the modelled modes, can then be recombined using equations (6) and (7).

## 3. CASES WITH INTERNAL RESONANCES

Non-linear systems with internal resonances can arise fairly frequently in structural dynamics. Typically, in such cases, the dynamic equations of the system feature two or more modes with non-removable coupling terms that excite each other at or near resonance, even without external forcing. The analytical treatment of such systems in itself is beyond the scope of this study (see, for example, references [8, 9]).

For complex structural systems with many degrees of freedom, seeking potential internal resonances and, subsequently, analyzing them, can become very impractical. From an engineering viewpoint, the alternative consisting in simulating the system numerically is very attractive but, unfortunately, as alluded to earlier, the commonly employed linear modal analysis of non-linear systems may completely overlook the occurrence of internal resonances, forcing the engineer to anticipate (or ignore) such occurrences. Also, one may end up using many more modes than are actually required to capture the dynamics of interest.

By contrast, the non-linear modal analysis developed in section 2 intrinsically captures this phenomenon without requiring prior knowledge of its existence: internal resonances involving only modelled modes are automatically accounted for by the procedure (via the coupled dynamic equations on the multi-mode manifold—equation (8)), while internal resonances between modelled and non-modelled modes are signalled by singularities in the Taylor series expansions of the manifold. More precisely, if the $j$th non-modelled mode is resonant with one (or more) of the modelled modes, some of the coefficients in the expansions of $X_j$ and $Y_j$ (equations (6) and (7)) will be singular (namely, the coefficients corresponding to the coupling terms between the resonant modes). These singularities are caused by the presence of the non-removable coupling terms between the modelled and non-modelled resonant modes, which is in direct contradiction with the concept of invariance of the multi-mode manifold (dynamic interactions between the manifold and the resonant non-modelled mode would be necessary to recover the coupling, but are prevented by the formulation of the problem in terms of an invariant manifold—hence the singularities). The generation of a multi-mode invariant manifold which does not include the $j$th mode is thus bound to fail from a mathematical viewpoint, which is desirable from a physical viewpoint. The remedy is, of course, to include the non-modelled, yet resonant, mode into the set of modelled modes, in which case the internal resonance is readily treated.

As an aside, it should be noted that if one wishes to analyze a particular internal resonance in a system with many degrees of freedom using traditional analytical techniques, this multi-mode methodology provides the proper way of obtaining the desired reduced set of coupled dynamic equations (by letting $S_m$ be the subset of the resonant modes only). The analysis can then be performed on the resulting equations, with the consequences on the whole system being recovered from equations (6) and (7) (see reference [10]).

It should also be realized that this methodology cannot detect all possible internal resonances between modelled and non-modelled modes, due to built-in restrictions brought by the Taylor series expansions, but that it will exhibit all the most important ones. In fact, each monomial in the series expansions (equations (6) and (7)) only involves a limited number of modes (the number of different modelled modes involved in that particular monomial plus the mode $j \notin S_m$ under consideration; i.e., at most, the degree of the monomial plus one). This implies that first order (linear) determinations of multi-mode manifolds will only detect two-mode internal resonances with non-modelled modes (if they exist), regardless of the number of modelled modes, second order (quadratic) determinations will only detect two- or three-mode internal resonances with non-modelled modes, etc. Consequently, while internal resonances between *any* number of modelled modes are automatically accounted for by the procedure, some internal resonances between modelled and non-modelled modes may not be detected. However, when $M$ modes are modelled, and when an approximation at order $N$ of the corresponding manifold is sought, all relevant internal resonances between modelled and non-modelled modes involving up to $\min(N+1, M+1)$ modes will be detected. For instance, a cubic order approximation of an invariant manifold can detect important internal resonances with non-modelled modes, involving up to four modes if at least three modes are modelled, or up to the number of modelled modes plus one otherwise, which is sufficient for most applications in structural dynamics.

Finally, it may be noted, for the sake of completeness, that internal resonances between non-modelled modes are of no consequence, since they do not violate the invariance property and the modes involved remain unexcited for all time.

## 4. CONCLUSIONS

The non-linear modal analysis introduced herein is thought to have potentially important applications for the dynamic analysis of structural and other non-linear systems.

It allows one to perform a legitimate modal analysis of the free response of non-linear systems, based on specific characteristics of such systems and on a geometric approach, and recovers some desirable properties of the linear modal analysis of linear systems. In particular, this methodology leaves the modelled modes invariant from the non-modelled ones (and vice versa), so that a reduced set of equations only can be simulated. On the other hand, interactions between the modelled modes are accounted for, and for instance, internal resonances between them are treated without additional work.

In the case of internal resonances between modelled and non-modelled modes, the procedure exhibits some singularities which point at the resonant modes. Care should then be taken to incorporate all necessary modes in the set of the modelled modes. This should not be regarded as a potential case in which the non-linear modal analysis procedure fails, but rather as a feature revealing an erroneous original choice of modelled modes.

This approach has been applied to non-linear structural systems with internal resonances. Numerical results show that reduced order models based on non-linear modal analysis require far fewer modes than standard linear modal analysis to achieve a desired accuracy. These results are currently being written into a full-length paper (see reference [11]).

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