ME/ECE859-Spring 2008 Homework 2, Due date: 1/30/08 Wed

The spread of infective diseases can be modeled as a nonlinear system. We introduce here the SIRS model. The population consists of three disjoint groups. The population of susceptible individuals is denoted by S, the infected population by I, and the recovered population by R. We assume that the total population, which is denoted by τ is constant, so that $\frac{d}{dt}(\tau = S + I + R) = 0$. We assume that the rate of transmission of the disease (denoted by β) is proportional to the number of encounters between susceptible and infected individuals. Also assume that the return of recovered individuals to the class S occurs at a rate (denoted by μ) proportional to the population of recovered individuals (like malaria and tuberculosis). Finally assume that the rate at which infected individuals recovers (denoted by ν) is proportional to the number of infected. Hence the SIRS model is given by

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI + \mu R \\ \frac{dI}{dt} &= \beta SI - \nu I \\ \frac{dR}{dt} &= \nu I - \mu R, \end{aligned}$$

where β, μ, ν and τ are positive real numbers. Since τ is constant, we eliminate R from this system by using $R = \tau - S - I$:

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI + \mu(\tau - S - I) \\ \frac{dI}{dt} &= \beta SI - \nu I. \end{aligned} \tag{1}$$

Let the domain of interest be $D := \{(S, I) \mid S \geq 0, I \geq 0, S + I \leq \tau\}$. Consider the state equation (1) over D.

1. Find all possible equilibria and determine the type of each one.

2. Study bifurcation as τ varies and assume that other parameters are fixed. Discuss the type of bifurcation and sketch the bifurcation diagram. Obtain different phase portraits to support your bifurcation diagram using some numerical numbers for parameters. Explain the obtained bifurcation diagram in terms of the disease spread.

3. Repeat 2. by varying ν and assuming that other parameters are fixed.

4. Prove that the region of interest D is *positively invariant*[1] by checking the direction of the vector field along the boundary of D.

[1] A set D is said to be a *positively invariant* set if $x(0) \in D \Rightarrow x(t) \in D$, $\forall t \geq 0$.