1. Set notation
2. Fields, vector spaces, normed vector spaces, inner product spaces
3. More notation
4. Vectors in $\mathbf{R}^n, \mathbf{C}^n$, norms
5. Matrix Facts (determinants, inversion formulae)
6. Normed vector spaces, inner product spaces
7. Linear transformations
8. Matrices, matrix multiplication as linear transformation
9. Induced norms of matrices
10. Schur decomposition of matrices
11. Symmetric, Hermitian and Normal matrices
12. Positive and Negative definite matrices
13. Singular Value decomposition
14. Hermitian square roots of positive semidefinite matrices
15. Schur complements
16. Matrix Dilation, Parrott's theorem
17. Completion of Squares

1. $\mathbf{R}$ is the set of real numbers. $\mathbf{C}$ is the set of complex numbers.
2. $\mathbf{N}$ is the set of integers.
3. The set of all $n \times 1$ column vectors with real number entries is denoted $\mathbf{R}^n$. The i'th entry of a column vector x is denoted x_i.
4. The set of all $n \times m$ rectangular matrices with complex number entries is denoted $\mathbf{C}^{n \times m}$. The element in the i'th row, j'th column of a matrix M is denoted by M_{ij}, or m_{ij}.
5. Set notation:
   (a) $a \in A$ is read: "a is an element of A"
   (b) $X \subset Y$ is read: "X is a subset of Y"
   (c) If A and B are sets, then $A \times B$ is a new set, consisting of all ordered-pairs drawn from A and B,
   $$A \times B := \{(a, b) : a \in A, b \in B\}$$
   (d) The expression $\{\mathcal{A} : \mathcal{B}\}$ is read as:
   "The set of all `insert expression` $\mathcal{A}$
   such that `insert expression` $\mathcal{B}$."
   Hence
   $$\left\{ x \in \mathbf{R}^3 : \sum_{i=1}^{3} x_i^2 \leq 1 \right\}$$
   is the ball of radius 1, centered at the origin, in 3-dimensional euclidean space.
6. The notation $f : X \to Y$ implies that X and Y are sets, and f is a function mapping X into Y

Fields

A *field* consists of: a set $\mathcal{F}$ (which must contain at least 2 elements) and two operations, *addition* $(+)$ and *multiplication* $(\cdot)$, each mapping $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$. Several axioms must be satisfied:

- For every $a, b \in \mathcal{F}$, there corresponds an element $a + b \in \mathcal{F}$, *the addition of* a *and* b. For all $a, b, c \in \mathcal{F}$, it must be that

$$
\begin{aligned}
a + b &= b + a \\
(a + b) + c &= a + (b + c)
\end{aligned}
$$

- There is a unique element $\theta \in \mathcal{F}$ (or $0_{\mathcal{F}}$, $\theta_{\mathcal{F}}$, or just 0) such that for every $a \in \mathcal{F}, a + \theta = a$. Moreover, for every $a \in \mathcal{F}$, there is a unique element labled $-a$ such that $a + (-a) = \theta$.

- For every $a, b \in \mathcal{F}$, there corresponds an element $a \cdot b \in \mathcal{F}$, *the multiplication of* a *and* b. For every $a, b, c \in \mathcal{F}$

$$
\begin{aligned}
a \cdot b &= b \cdot a \\
a \cdot (b \cdot c) &= (a \cdot b) \cdot c.
\end{aligned}
$$

- There is a unique element $1_{\mathcal{F}} \in \mathcal{F}$ (or just 1) such that for every $a \in \mathcal{F}, 1 \cdot a = a \cdot 1 = a$. Moreover, for every $a \in \mathcal{F}, a \neq \theta$, there is a unique element, labled $a^{-1} \in \mathcal{F}$ such that $a \cdot a^{-1} = 1_{\mathcal{F}}$.

- For every $a, b, c \in \mathcal{F}$,

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Example: The real numbers $\mathbf{R}$, the complex numbers $\mathbf{C}$, and the rational numbers $\mathbf{Q}$ are three examples of fields.

Vector Spaces

A vector space consists of:

- a set $\mathcal{V}$, whose elements are called "vectors," and
- a field $\mathcal{F}$ (often just $\mathbf{R}$ or $\mathbf{C}$, and then denoted $\mathbf{F}$) whose elements are "scalars."

Two operations,

- *addition of vectors*, and
- *scalar multiplication*

are defined and must satisfy the following relationships:

- For every $u, w \in \mathcal{V}$, there corresponds a vector $u + w \in \mathcal{V}$ such that for all $u, v, w \in \mathcal{V}$

  1. $u + w = w + u$
  2. $(u + w) + v = u + (w + v)$

  There is a unique vector $\theta_{\mathcal{V}}$ (or $0_{\mathcal{V}}$, θ, or just 0) such that for every $w \in \mathcal{V}, w + \theta_{\mathcal{V}} = w$. Moreover, for every $w \in \mathcal{V}$, there is a unique vector labled $-w$ such that $w + (-w) = \theta_{\mathcal{V}}$.

- For every $\alpha \in \mathbf{F}$ and $w \in \mathcal{V}$ there corresponds a vector $\alpha w \in \mathcal{V}$. The operation must satisfy $1w = w$ for all $w \in \mathcal{V}$ and for every $u, w \in \mathcal{V}, \alpha, \beta \in \mathbf{F}$ the distributive laws

  1. $\alpha(u + w) = \alpha u + \alpha w$
  2. $(\alpha + \beta)u = \alpha u + \beta u$

  must hold.

If Z and W are vector spaces over the same $\mathcal{F}$, then $Z \times W$ is also a vector space (field $\mathcal{F}$), with addition and scalar multiplication defined "coordinatewise."

Specifically, if $q_1, q_2 \in Z \times W$, then each q_i is of the form

$$q_i = (z_i, w_i).$$

For $\alpha \in \mathcal{F}$, define

$$\alpha q_1 := (\alpha z_1, \alpha w_1), \qquad q_1 + q_2 := (z_1 + z_2, w_1 + w_2)$$

- $n > 0$, $\mathcal{V} = \mathbf{R}^n$, $\mathcal{F} = \mathbf{R}$, addition and scalar multiplication defined in terms of components

$$(x+y)_i := x_i + y_i, \quad (\alpha x)_i := \alpha x_i$$

- $n > 0$, $\mathcal{V} = \mathbf{C}^n$, $\mathcal{F} = \mathbf{C}$, addition and scalar multiplication again defined in terms of components.
- $n > 0$, $\mathcal{V} = \mathbf{C}^n$, $\mathcal{F} = \mathbf{R}$, addition and scalar multiplication again defined in terms of components.
- $n, m > 0$, $\mathcal{V} = \mathbf{F}^{n \times m}$, $\mathcal{F} = \mathbf{F}$, addition and scalar multiplication defined entrywise

$$(A+B)_{i,j} := A_{i,j} + B_{i,j}, \quad (\alpha A)_{i,j} := \alpha A_{i,j}$$

- $\mathcal{V} :=$ all continuous, real $-$ valued functions defined on $[0\ 1], \mathcal{F} = \mathbf{R}$. Addition and scalar multiplication defined pointwise: for $f, g \in \mathcal{V}, \alpha \in \mathbf{R}$

$$(f+g)(x) := f(x) + g(x), \quad (\alpha f)(x) := \alpha f(x)$$

- $\mathcal{V} :=$ all piecewise continuous, real-valued functions defined on $[0\ \infty)$, with a finite number of discontinuities in any finite interval, $\mathcal{F} = \mathbf{R}$. Addition and scalar multiplication defined pointwise, as before. For future, call this space $\mathrm{PC}[0, \infty)$.
- Same function space as above, with further restriction that

$$\max_{x \geq 0} |f(x)| < \infty \qquad \text{or} \qquad \int_0^\infty |f(\eta)|\, d\eta < \infty$$

Call these $\mathrm{PC}_\infty[0, \infty)$, and $\mathrm{PC}_1[0, \infty)$, respectively.

1. In a statement, if $\mathbf{F}$ appears, it means that the statement is true with $\mathbf{F}$ replaced by either $\mathbf{R}$ or $\mathbf{C}$ throughout the statement.
2. The set of all $n \times 1$ column vectors with real number entries is denoted $\mathbf{R}^n$.
3. The set of all $n \times m$ rectangular matrices with complex number entries is denoted $\mathbf{C}^{n\times m}$. The element in the i'th row, j'th column of a matrix M is denoted by M_{ij}, or m_{ij}.
4. If $x \in \mathbf{C}$, $\bar{x} \in \mathbf{C}$ is the complex conjugate of x.
5. If $M \in \mathbf{F}^{n\times m}$, then M^T is the transpose of M; M^* is the complex-conjugate transpose of M
6. If $Q \in \mathbf{F}^{n\times n}$, and $Q^*Q = I_n$, then Q is called *unitary.*
7. $\mathbf{R}_+ := \{\alpha \in \mathbf{R} : \alpha \geq 0\}$, $\mathbf{N}_+ := \{k \in \mathbf{N} : k \geq 0\}$

1. Eigenvalues: $\lambda \in \mathbf{C}$ is an *eigenvalue* of $M \in \mathbf{F}^{n\times n}$ if there is a vector $v \in \mathbf{C}^n, v \neq 0_n$, such that

$$Mv = \lambda v$$

   The vector v is called *an eigenvector* associated with eigenvalue λ.

2. The eigenvalues of $M \in \mathbf{F}^{n\times n}$ are the roots of the equation

$$p_M(\lambda) := \det(\lambda I_n - M) = 0$$

3. **Fact:** Every matrix has at least one eigenvalue and associated eigenvector, since the polynomial $p_M(\lambda)$ has at least one root.

4. **Fact:** The eigenvalues of a matrix are continuous functions of the entries of the matrix

5. For any $n \times m$ matrix A, and $m \times n$ matrix B, the nonzero eigenvalues of AB are equal to the nonzero eigenvalues of BA.

6. A matrix $M \in \mathbf{F}^{n\times n}$ is called *Hurwitz* if all of its eigenvalues have negative real parts.

7. A matrix $M \in \mathbf{F}^{n\times n}$ is called *Schur* if all of its eigenvalues have absolute value less than 1.

1. If A and B are square matrices, then
   (a) $\det(AB) = \det(BA) = \det(A)\det(B)$
   (b) $\det(A) = \det(A^T)$
   (c) $\det(A^*) = \overline{\det(A)}$
2. For any $n \times m$ matrix A, and $m \times n$ matrix B,
   (a) $\det(I_n + AB) = \det(I_m + BA)$
   (b) $(I_n + AB)$ is invertible if and only if $(I_m + BA)$ is invertible, and moreover,
   (c) $(I_n + AB)^{-1} A = A (I_m + BA)^{-1}$
3. If X and Z are square, Y compatible, then

$$\det\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \det(X)\det(Z)$$

4. If X and Z are square, invertible, Y compatible, then

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}^{-1} = \begin{bmatrix} X^{-1} & 0 \\ -Z^{-1}YX^{-1} & Z^{-1} \end{bmatrix}$$

5. If A and D are square, D invertible, B, C compatible dimensions, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C & D \end{bmatrix}$$

so that

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det\left(A - BD^{-1}C\right)\det(D)$$

1. Suppose A and D are square, D invertible, B, C compatible dimensions. If $A - BD^{-1}C$ is invertible then

$$\begin{aligned}\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} &= \begin{bmatrix} I & 0 \\ -D^{-1}C & D^{-1} \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)^{-1} & -\left(A - BD^{-1}C\right)^{-1} BD^{-1} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} (A - BD^{-1}C)^{-1} & -\left(A - BD^{-1}C\right)^{-1} BD^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & D^{-1}C\left(A - BD^{-1}C\right)^{-1} BD^{-1} + D^{-1} \end{bmatrix}\end{aligned}$$

2. If A and D are square, invertible, B, C compatible dimensions, then

$$\det(D) \det\left(A - BD^{-1}C\right) = \det(A) \det\left(D - CA^{-1}B\right)$$

and if not 0, then

$$\left(A - BD^{-1}C\right)^{-1} = A^{-1} + A^{-1}B\left(D - CA^{-1}B\right)^{-1} CA^{-1}$$

3. If A is square and invertible, and B, C and D are compatibly dimensioned, then vectors d_1, d_2, e_1 and e_2 satisfy

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

if and only if they satisfy

$$\begin{bmatrix} d_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} e_1 \\ d_2 \end{bmatrix}$$

In reparametrizing some optimization problems involving feedback, the following is useful: Let $T \in \mathbf{F}^{n \times m}$ be given. Define

$$S_1 := \left\{ K\left(I - TK\right)^{-1} : K \in \mathbf{F}^{m \times n}, \det\left(I - TK\right) \neq 0 \right\}$$

$$S_2 := \left\{ Q \in \mathbf{F}^{m \times n} : \det\left(I - QT\right) \neq 0 \right\}$$

Then $S_1 = S_2$, and S_2 is dense in $\mathbf{F}^{m \times n}$; that is, for any $\tilde{Q} \in \mathbf{F}^{m \times n}$, and any $\epsilon > 0$, there is a $Q \in S_2$ such that

$$\max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left| \tilde{q}_{ij} - q_{ij} \right| < \epsilon$$

Suppose $(\mathcal{V}, \mathbf{F})$ is a vector space (again, $\mathbf{F}$ is either $\mathbf{R}$ or $\mathbf{C}$). If there is a function $\|\cdot\| : \mathcal{V} \to \mathbf{R}$ such that for any $u, v \in \mathcal{V}$, and $\alpha \in \mathbf{F}$

- $\|u\| \geq 0$
- $\|u\| = 0 \;\Leftrightarrow u = 0_n$
- $\|\alpha u\| = |\alpha| \, \|u\|$
- $\|u + v\| \leq \|u\| + \|v\|$

then the function $\|\cdot\|$ is called *a norm* on $\mathcal{V}$, and $(\mathcal{V}, \mathbf{F})$ is a *normed vector space*

For a vector $v \in \mathbf{F}^n$, let v_i be the i'th component. Define

$$\begin{aligned} \|v\|_1 &:= \sum_{i=1}^{n} |v_i| \\ \|v\|_2 &:= \left(\sum_{i=1}^{n} |v_i|^2 \right)^{1/2} \\ \|v\|_\infty &:= \max_{1 \leq i \leq n} |v_i| \end{aligned}$$

Each of these separate definitions satisfy all of the 4 axioms that a *norm* must satisfy (all axioms are easy to check except triangle inequality for $\|\cdot\|_2$, which we will verify in a few slides).

Hence each of $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ are norms on $\mathbf{F}^n$.

We will pretty much exclusively use the $\|\cdot\|_2$ norm and often drop the subscript 2, simply using $\|\cdot\|$. Some easy facts are

1. For $v \in \mathbf{F}^n$, $\|v\|^2 = v^* v$
2. For $v \in \mathbf{F}^n, w \in \mathbf{F}^m$, $\left\| \begin{array}{c} v \\ w \end{array} \right\|^2 = \|v\|^2 + \|w\|^2$.
3. If $Q \in \mathbf{F}^{n \times n}$, $Q^* Q = I_n$, then for all $v \in \mathbf{F}^n$, $\|Qv\| = \|v\|$
4. Given $Q \in \mathbf{F}^{n \times n}, Q^* Q = I_n$,

   $$\{x : x \in \mathbf{F}^n, \|x\| \leq 1\} = \{Qx : x \in \mathbf{F}^n, \|x\| \leq 1\}$$

   and

   $$\{x : x \in \mathbf{F}^n, \|x\| = 1\} = \{Qx : x \in \mathbf{F}^n, \|x\| = 1\}$$

A vector space $(\mathcal{V}, \mathbf{F})$ is an *inner product* space if there is a function $< \cdot, \cdot >$: $\mathcal{V} \times \mathcal{V} \rightarrow \mathbf{C}$ such that for every $u, v, w \in \mathcal{V}$ and $\alpha \in \mathbf{F}$ the following hold:

1. $\langle u, v \rangle = \overline{\langle v, u \rangle}$
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. $\langle \alpha u, w \rangle = \bar{\alpha} \langle u, w \rangle$
4. $\langle u, u \rangle \geq 0$
5. $\langle u, u \rangle = 0$ if and only if $u = \mathbf{0}$.

The function $\langle \cdot, \cdot \rangle$ is called the inner product on $\mathcal{V}$.

Two vectors $u, w \in \mathcal{V}$ are said to be *perpindicular*, written $u \perp w$ if $\langle u, w \rangle = 0$.

The most important inner product spaces that we will use in this section are $(\mathbf{R}^n, \mathbf{R})$ and $(\mathbf{C}^n, \mathbf{C})$, with inner products defined as

$$u, w \in \mathbf{R}^n, \ \langle u, w \rangle := \sum_i^n u_i w_i = u^T w$$

$$u, w \in \mathbf{C}^n, \ \langle u, w \rangle := \sum_i^n \bar{u}_i w_i = u^* w$$

On $(\mathcal{V}, \mathbf{F})$, define a function using by the inner-product. For each $v \in \mathcal{V}$ define

$$N(v) := \sqrt{\langle v, v \rangle}$$

The Schwarz inequality relates inner products and N.

Theorem: For each $u, w \in \mathcal{V}$ $|\langle u, w \rangle| \leq N(u)N(w)$.

Proof: Given u and w, find complex number α with $|\alpha| = 1$, and $\alpha\langle u, w \rangle = |\langle u, w \rangle|$. Then for any real number t,

$$0 \leq \langle u + t\alpha w, u + t\alpha w \rangle = N(u)^2 \ + \ 2t\, |\langle u, w \rangle| \ + t^2 N(w)^2.$$

This is a quadratic function. Characterizing that the minimum (over the real variable t) is non-negative gives the result.

$$|\langle u, w \rangle| \leq N(u)N(w)$$

The triangle inequality follows for N as well: Given any $u, w \in \mathcal{V}$,

$$\begin{aligned} N(u+w)^2 &= \langle u + w, u + w \rangle \\ &= N(u)^2 + 2\mathrm{Re}\left(\langle u, w \rangle\right) + N(w)^2 \\ &\leq N(u)^2 + 2\,|\langle u, w \rangle| + N(w)^2 \\ &\leq N(u)^2 + 2N(u)N(w) + N(w)^2 \\ &= \left(N(u) + N(w)\right)^2 \end{aligned}$$

Hence, N is actually a norm on $\mathcal{V}$, so every inner-product space is in fact a normed vector space, using N, the *norm induced from the inner product*. So, unless otherwise notated, using the symbol $\|\cdot\|$ when working with a inner-product space means the norm induced from the inner product.

Note, if u and w are perpindicular, then $\|u + w\|^2 = \|u\|^2 + \|w\|^2$, which is the "Pythagorean" theorem.

Take $A \in \mathbf{C}^{n\times m}$. Then

1. The m columns of $\begin{bmatrix} I_m \\ A \end{bmatrix}$ are linearly independent, and are perpindicular to the n linearly independent columns of $\begin{bmatrix} -A^* \\ I_n \end{bmatrix}$
2. Take $n > m$, and assume the columns of A are linearly independent. Suppose $A_\perp$ is $n \times (n-m)$, has linearly independent columns, and $A_\perp^* A = 0$. If X is $n \times n$, and invertible, then XA and $X^{-*}A_\perp$ each have linearly independent columns, and are perpindicular to one another.

Suppose $\mathcal{V}$ and $\mathcal{W}$ are vector spaces over the same field $\mathcal{F}$. If $\mathcal{L} : \mathcal{V} \to \mathcal{W}$ satisfies

$$\mathcal{L}(\alpha v + \beta u) = \alpha \mathcal{L}(v) + \beta \mathcal{L}(u)$$

for all $\alpha, \beta \in \mathcal{F}$, and all $v, u \in \mathcal{V}$, then $\mathcal{L}$ is a *linear transformation on* $\mathcal{V}$ *to* $\mathcal{W}$.

Examples:

1. $\mathcal{V} = \mathbf{C}^m$, $\mathcal{W} = \mathbf{C}^n$, $M \in \mathbf{C}^{n \times m}$, and $\mathcal{L}$ defined by matrix-vector multiplication: For $v \in \mathcal{V}$, define $\mathcal{L}(v)$ as

$$\mathcal{L}(v) := Mv, \qquad \text{or componentwise} \quad (\mathcal{L}(v))_i := \sum_{j=1}^{m} M_{ij} v_j$$

2. $\mathcal{V} = \mathbf{R}^{n \times n}$, $\mathcal{W} = \mathbf{R}^{n \times n}$, $A \in \mathbf{R}^{n \times n}$, and $\mathcal{L}$ defined by a Lyapunov operator, For $P \in \mathcal{V}$, define $\mathcal{L}(P)$ as

$$\mathcal{L}(P) := A^T P + PA$$

3. $\mathcal{V} = \mathrm{PC}_\infty[0, \infty)$, $\mathcal{W} = \mathrm{PC}_\infty[0, \infty)$, $g \in \mathrm{PC}_1[0, \infty)$, and $\mathcal{L}$ defined by convolution, For $v \in \mathcal{V}$, define $\mathcal{L}v$ as

$$(\mathcal{L}v)(t) := \int_0^t g(t - \tau) v(\tau) d\tau$$

For the remainder of this handout, focus on the linear operator defined by matrix-vector multiplication, and other results about matrices.

If $M \in \mathbf{F}^{n\times m}$, then M naturally defines a linear transformation $\mathcal{L}_M : \mathbf{F}^m \to \mathbf{F}^n$ via standard matrix-vector multiplication.

For any $v \in \mathbf{R}^m$

$$\mathcal{L}_M(v) := Mv$$

Typically, we will not take care to distingush the matrix from the operation. Simply note that matrix-vector multiplication in a linear transformation on the vector, namely, for all $u, v \in \mathbf{F}^m$, $\alpha, \beta \in \mathbf{F}$,

$$M(\alpha u + \beta v) = \alpha M u \ + \ \beta M v$$

Using norms in $\mathbf{F}^m$ and $\mathbf{F}^n$, the norm of the matrix transformation can be characterized

Define

$$\|M\|_{\alpha\leftarrow\beta} := \max_{u\in\mathbf{F}^m, u\neq 0_m} \frac{\|Mu\|_\alpha}{\|u\|_\beta}$$

This is the maximum amplification obtainable, via matrix-vector multiplication, measuring sizes in the domain and range with norms.

Easy Facts: For $M \in \mathbf{F}^{n \times m}$,

1. Other characterizations are possible
$$\|M\|_{\alpha \leftarrow \beta} = \max_{u \in \mathbf{R}^m, \|u\|_\beta \leq 1} \|Mu\|_\alpha = \max_{u \in \mathbf{R}^m, \|u\|_\beta = 1} \|Mu\|_\alpha$$
2. Easily proven: $\|M\|_{1 \leftarrow 1} = \max_{1 \leq j \leq m} \sum_{i=1}^{n} |M_{ij}|$
3. Easily proven: $\|M\|_{\infty \leftarrow \infty} = \max_{1 \leq i \leq n} \sum_{j=1}^{m} |M_{ij}|$
4. Later: $\|M\|_{2 \leftarrow 2}$ is characterized in terms of the eigenvalues of M^*M.
5. Interchanging rows and/or columns of M does not change $\|M\|_{1 \leftarrow 1}$, $\|M\|_{2 \leftarrow 2}$, or $\|M\|_{\infty \leftarrow \infty}$.
6. Given $U \in \mathbf{F}^{n \times n}, V \in \mathbf{F}^{m \times m}$ both unitary (ie., $U^*U = I_n, V^*V = I_m$), then for any $M \in \mathbf{F}^{n \times m}$,
$$\|UMV\|_{2 \leftarrow 2} = \|M\|_{2 \leftarrow 2}$$
7. If $\|M\|_{\alpha \leftarrow \alpha} < 1$, then $\det(I - M) \neq 0$
8. For matrices A, B, C of appropriate dimensions,
$$\|AB\|_{\alpha \leftarrow \gamma} \leq \|A\|_{\alpha \leftarrow \beta} \|B\|_{\beta \leftarrow \gamma}$$
$$\|A + C\|_{\alpha \leftarrow \gamma} \leq \|A\|_{\alpha \leftarrow \gamma} + \|C\|_{\alpha \leftarrow \gamma}$$
9. Deleting rows and/or columns does not increase $\|\cdot\|_{p \leftarrow p}$. Specifically, for matrices A, B, C of appropriate dimensions,
$$\left\| \begin{bmatrix} A & B \end{bmatrix} \right\|_{p \leftarrow p} \geq \|A\|_{p \leftarrow p}, \qquad \left\| \begin{bmatrix} A \\ C \end{bmatrix} \right\|_{p \leftarrow p} \geq \|A\|_{p \leftarrow p}$$

Theorem: Given a matrix $A \in \mathbf{C}^{n \times n}$. There exists a matrix $Q \in \mathbf{C}^{n \times n}$ with

- $Q^*Q = I_n$, and
- $Q^*AQ =: \Lambda$ upper triangular.

Remarks:

1. Proof is straightforward – induction along with Gram-Schmidt Orthonormalization process.
2. The matrix Q has orthonormal rows and columns (since $Q^*Q = QQ^* = I_n$)
3. Since Q^*AQ is upper triangular, the eigenvalues of Q^*AQ are the diagonal entries.
4. In this case, $Q^{-1} = Q^*$, so the eigenvalues of Q^*AQ are the same as the eigenvalues of A. The order that the eigenvalues appear is arbitrary (they can be sorted in any order). This will be clear from the proof.
5. The Matlab command `schur` computes (reliably and quickly) a Schur decomposition.

Note that the theorem is true for 1×1 matrices, ie., $n = 1$, simply take $Q := 1$, and $\Lambda = A$.

Now, suppose that the theorem statement is true for $n = k$, ie., suppose it is true for $k \times k$ matrices. Furthermore, let $A \in \mathbf{F}^{(k+1)\times(k+1)}$. Let $v \in \mathbf{C}^{k+1}$ be an eigenvector of A, with corresponding eigenvalue $\lambda \in \mathbf{C}$ (possible since every matrix has at least one eigenvalue). By definition, $v \neq 0_{k+1}$, and hence we can (by dividing) assume that $v^* v = 1$. Now, using the Gram-Schmidt orthogonalization procedure, choose vectors $v_1, v_2, \ldots, v_k$ each in $\mathbf{C}^{k+1}$ such that

$$\{v, v_1, v_2, \ldots, v_k\}$$

is a set of mutually orthonormal vectors. Stack these into a square, $(k+1) \times (k+1)$ matrix $V := [v\ v_1\ v_2\ \cdots\ v_k]$.

Note that $V^* V = I_{k+1}$. Moreover, there is a matrix $\Gamma \in \mathbf{C}^{k \times k}$, and a vector $w \in \mathbf{C}^k$ such that

$$AV = V \begin{bmatrix} \lambda & w^* \\ 0 & \Gamma \end{bmatrix}$$

By then induction hypothesis, since Γ is of dimension k, there is a matrix $P \in \mathbf{C}^{k \times k}$ and upper triangular $\Psi \in \mathbf{C}^{k \times k}$ with $P^* P = I_k$ and $P^* \Gamma P = \Psi$. Hence, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & P^* \end{bmatrix} V^* A V \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & P^* \end{bmatrix} \begin{bmatrix} \lambda & w^* \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} = \begin{bmatrix} \lambda & w^* P \\ 0 & \Psi \end{bmatrix}$$

which is indeed upper triangular. Moreover

$$Q := V \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$$

has $Q^* Q = I_{k+1}$ as desired. $\sharp$

Definition: The set of real, symmetric $n \times n$ matrices is denoted $\mathcal{S}^{n\times n}$, and defined as

$$\mathcal{S}^{n\times n} := \left\{ M \in \mathbf{R}^{n\times n} : M^T = M \right\}$$

Definition: The set of complex, Hermitian $n \times n$ matrices is denoted $\mathcal{H}^{n\times n}$, and defined as

$$\mathcal{H}^{n\times n} := \{ M \in \mathbf{C}^{n\times n} : M^* = M \}$$

Definition: The set of complex, normal $n \times n$ matrices is denoted $\mathcal{N}^{n\times n}$, and defined as

$$\mathcal{N}^{n\times n} := \{ M \in \mathbf{C}^{n\times n} : M^* M = M M^* \}$$

Note that

$$\mathcal{S}^{n\times n} \subset \mathcal{H}^{n\times n} \subset \mathcal{N}^{n\times n}$$

Fact: Hermitian matrices have real eigenvalues:

Proof: Let $\lambda \in \mathbf{C}$ be an eigenvalue of a Hermitian matrix $M = M^*$, and let $v \neq 0_n$ be a corresponding eigenvector, so that $Mv = \lambda v$.

Note that

$$\begin{aligned} 2\mathrm{Re}\,(\lambda)\,\|v\|^2 &= \lambda\,\|v\|^2 + \bar{\lambda}\,\|v\|^2 \\ &= v^*(\lambda v) + (\lambda v)^* v \\ &= v^* M v + (Mv)^*\, v \\ &= v^* M v + v^* M^* v \\ &= v^* M v + v^* M v \qquad \text{using } M = M^* \\ &= 2 v^* M v \\ &= 2\lambda\,\|v\|^2 \end{aligned}$$

Since $v \neq 0_n$, the norm is positive, divide out leaving

$$\mathrm{Re}\,(\lambda) = \lambda$$

as desired.

Remark: If $M \in \mathcal{H}^{n \times n}$, the eigenvalues of M are real, and can be ordered

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

and it makes sense to write

$$\lambda_{\max}(M) \quad \text{and} \quad \lambda_{\min}(M)$$

without confusion

Fact: An upper triangular, normal matrix is actually diagonal.

Check it out...

Fact: Given $Q \in \mathbf{C}^{n \times n}$ satisfying $Q^*Q = I_n$, then for any $M \in \mathbf{C}^{n \times n}$,

$$M \in \mathcal{N} \Leftrightarrow Q^*MQ \in \mathcal{N}$$

The proof is simple:

$$\begin{aligned} M^*M = MM^* & \leftrightarrow Q^*\left(M^*M\right)Q = Q^*\left(MM^*\right)Q \\ & \leftrightarrow Q^*M^*MQ = Q^*MM^*Q \\ & \leftrightarrow Q^*M^*\underbrace{QQ^*}_{I}MQ = Q^*M\underbrace{QQ^*}_{I}M^*Q \\ & \leftrightarrow Q^*M^*QQ^*MQ = Q^*MQQ^*M^*Q \\ & \leftrightarrow \left(Q^*MQ\right)^*Q^*MQ = Q^*MQ\left(Q^*MQ\right)^* \end{aligned}$$

Hence,

Fact: A normal matrix M has an orthonormal set of eigenvectors, ie., there exists a matrices $Q, \Lambda \in \mathbf{C}^{n \times n}$ with

- $Q^*Q = I_n$,
- Λ diagonal
- $M = Q\Lambda Q^*$

If $M = M^*$, then

$$\{x^* M x : \|x\|_2 = 1\} = [\lambda_{\min}(M), \lambda_{\max}(M)]$$

Proof: Basic idea:

- Let $Q\Lambda Q^* = M$ be a Schur decomposition of M
- Since $M = M^*$, Λ is diagonal and real
- Notate $\xi := Q^* x$, noting $\|Q\xi\|_2 = \|\xi\|_2$ for all ξ,

Then

$$\begin{aligned}\{x^* M x : \|x\|_2 = 1\} &= \{x^* Q\Lambda Q^* x : \|x\|_2 = 1\} \\ &= \{\xi^* \Lambda \xi : \|Q\xi\|_2 = 1\} \\ &= \{\xi^* \Lambda \xi : \|\xi\|_2 = 1\} \\ &= \left\{\Sigma_{i=1}^n \lambda_i |\xi_i|^2 : \Sigma_{i=1}^n |\xi_i|^2 = 1\right\}\end{aligned}$$

For any $\alpha \in [0, 1]$, define

$$\xi_1 := \sqrt{\alpha},\ \xi_2 = \xi_3 = \cdots = \xi_{n+1} = 0,\ \xi_n := \sqrt{1-\alpha}$$

yielding

$$\sum_{i=1}^n \lambda_i |\xi_i|^2 = \alpha\lambda_1 + (1-\alpha)\lambda_n$$

which shows by proper choice of α, anything in between λ_1 and λ_n can be achieved.

Warning: Take $M = M^*$. Then

$$\{x^* M x : \|x\|_2 \leq 1\} \neq [\lambda_{\min}(M), \lambda_{\max}(M)]$$

Now, return to expression for $\|M\|_{2\leftarrow 2}$.

$$\begin{aligned} \|M\|_{2\leftarrow 2}^2 &:= \max_{\|x\|\leq 1} \|Mx\|^2 \\ &= \max_{\|x\|=1} \|Mx\|^2 \\ &= \max_{\|x\|=1} x^* M^* M x \\ &= \lambda_{\max}(M^* M) \end{aligned}$$

Hence, $\|M\|_{2\leftarrow 2}$ is often denoted by $\bar{\sigma}(M)$, called *the maximum singular value of* M. Since the nonzero eigenvalues of AB equal the nonzero eigenvalues of BA, it follows that

$$\bar{\sigma}(M) = \bar{\sigma}(M^*)$$

Definition: A matrix $M \in \mathcal{H}^{n\times n}$ is

1. *positive definite* (denoted $M \succ 0$) if $u^*Mu > 0$ for every $u \in \mathbf{C}^n, u \neq 0_n$.
2. *positive semi-definite* (denoted $M \succeq 0$) if $u^*Mu \geq 0$ for every $u \in \mathbf{C}^n$.
3. *negative definite* (denoted $M \prec 0$) if $u^*Mu < 0$ for every $u \in \mathbf{C}^n, u \neq 0_n$.
4. *negative semi-definite* (denoted $M \preceq 0$) if $u^*Mu \leq 0$ for every $u \in \mathbf{C}^n$.

For $A, B \in \mathcal{H}^{n\times n}$, write $A \preceq B$ if $A - B \preceq 0$. Similarly for $\prec, \succ$ and $\succeq$.

Easy Facts:

1. If $A \preceq B$ and $B \preceq A$, then indeed, $A = B$. If $A \preceq B$ and $C \preceq D$, then $A + C \preceq B + D$.
2. $L \in \mathbf{F}^{n\times n}$ invertible, $M \in \mathcal{H}^{n\times n}$, then
$$M \succ 0 \;\Leftrightarrow\; L^*ML \succ 0$$
3. $L \in \mathbf{F}^{n\times m}$ full column rank (so $n \geq m$), $M \in \mathcal{H}^{n\times n}$, then
$$M \succ 0 \;\Rightarrow\; L^*ML \succ 0$$
4. For any $W \in \mathbf{F}^{n\times m}$, $W^*W \succeq 0$.
5. For any $W \in \mathbf{F}^{n\times m}$, if $\text{rank}W = m$, then $W^*W \succ 0$.
6. $M \succ 0$ if and only if $\lambda_{\min}(M) > 0$.

7. If $M \in \mathcal{H}^{n \times n}$, then $M \prec 0 \quad \Leftrightarrow \quad (-M) \succ 0$

8. If $A_1, A_2 \in \mathcal{H}^{n \times n}$, $A_1 \succ 0$, $A_2 \succ 0$, then for each $t \in [0,1]$,

$$(1-t)A_1 + tA_2 \succ 0$$

9. Given $X \in \mathcal{H}^{n \times n}, Z \in \mathcal{H}^{m \times m}$ and $Y \in \mathbf{F}^{n \times m}$,

$$\begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \succ 0 \;\Rightarrow\; X \succ 0, Z \succ 0$$

10. $\bar{\sigma}(\cdot)$ bounds are easily converted into definiteness relations. For any matrix $M \in \mathbf{C}^{n \times m}$,

$$\begin{aligned} \bar{\sigma}(M) < \beta \;&\Leftrightarrow\; M^*M - \beta^2 I_m \prec 0 \\ &\Leftrightarrow\; MM^* - \beta^2 I_n \prec 0 \\ &\Leftrightarrow\; \bar{\sigma}(M^*) < \beta \end{aligned}$$

11. If M is invertible, and $M^* = M$, then $M \succ 0$ if and only if $M^{-1} \succ 0$.

12. **Warning:** If $M \neq M^*$, then M having positive, real eigenvalues does not guarantee $x^*Mx > 0$. Instead, check $M + M^*$, since it is Hermitian, and $x^*Mx = \frac{1}{2}x^*(M + M^*)x$. For example,

$$M = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$$

13. If $M + M^* \prec 0$, then eigenvalues of M have negative real-part

14. If $M = M^* \prec 0$, then for any $\Delta = \Delta^*$, there is an $\epsilon > 0$ such that $M + t\Delta \prec 0$ for all $|t| < \epsilon$.

Theorem: Let $T_{i\,i=0}^{k}$ be a family of matrices, with each $T_i \in \mathbf{C}^{n\times n}$, and $T_i^* = T_i$. If there exist scalars $\{d_i\}_{i=1}^k$ with $d_i \geq 0$, and

$$T_0 - \sum_{i=1}^{k} d_i T_i \succ 0$$

then for all $x \in \mathbf{C}^n$ which satisfy $x^* T_i x > 0$ for $1 \leq i \leq k$, it follows that $x^* T_0 x > 0$.

Proof: Let $x \in \mathbf{C}^n$ satisfy $x^* T_i x > 0$ for all $1 \leq i \leq k$. Hence, $x \neq 0$. By hypothesis, we have

$$x^* \left[T_0 - \sum_{i=1}^{k} d_i T_i \right] x > 0$$

which implies

$$x^* T_0 x > \sum_{i=1}^{k} d_i x^* T_i x \geq 0$$

as desired. $\sharp$

Remark: Easily replace $>$ with $\geq$ in above statement.

Theorem: Given $M \in \mathbf{F}^{n \times m}$. Then there exists

- $U \in \mathbf{F}^{n \times n}$, with $U^*U = I_n$,
- $V \in \mathbf{F}^{m \times m}$, with $V^*V = I_m$,
- integer $0 \leq k \leq \min(n, m)$, and
- real numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$

such that

$$M = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*$$

where $\Sigma \in \mathbf{R}^{k \times k}$ is

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix}$$

Proof: Clearly $M^*M \in \mathcal{H}^{m\times m}$ is positive semi-definite. Since it is Hermitian, it has a full set of orthonormal eigenvectors, and the eigenvalues are real, and nonnegative. Let $\{v_1, v_2, \ldots, v_m\}$ denote an orthonormal choice of eigenvectors, associated with the eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_m = 0$$

For any $1 \leq j \leq m$, we have

$$\begin{aligned} \|Mv_j\|^2 &= v_j^* M^* M v_j \\ &= \lambda_j v_j^* v_j \\ &= \lambda_j \end{aligned}$$

Hence, for $j > k$, it follows that $Mv_j = 0_n$.

For $1 \leq j \leq k$, define $\sigma_j := \sqrt{\lambda_j}$. Next, for $1 \leq j \leq k$, define vectors $u_j \in \mathbf{F}^n$ via

$$u_j := \frac{1}{\sigma_j} M v_j$$

Note that for any $1 \leq j, h \leq k$,

$$\begin{aligned} u_h^* u_j &= \tfrac{1}{\sigma_h \sigma_j} v_h^* M^* M v_j \\ &= \tfrac{1}{\sigma_h \sigma_j} v_h^* (\lambda_j v_j) \\ &= \tfrac{\sigma_j}{\sigma_h} v_h^* v_j \end{aligned}$$

This implies that $u_h^* u_j = \delta_{hj}$. Hence the set $\{u_1, \ldots, u_k\}$ are mutually orthonormal vectors in $\mathbf{F}^n$. Using Gram-Schmidt, construct vectors $u_{k+1}, \ldots, u_n$ to fill this out, so

$$\{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}$$

is a mutually orthonormal set if vectors in $\mathbf{F}^n$. Now we want to consider $u_h^* M v_j$ for 4 cases (depending on how h, j compare to k.

- $1 \leq h \leq k$ and $1 \leq j \leq k$. Substituting gives

$$\begin{aligned} u_h^* M v_j &= \tfrac{1}{\sigma_h} v_h^* M^* M v_j \\ &= \tfrac{\sigma_j}{\sigma_h} v_h^* v_l \\ &= \sigma_h \delta_{hj} \end{aligned}$$

- any h, with $j > k$. Substituting gives

$$\begin{aligned} u_h^* M v_j &= u_h^* (M v_j) \\ &= u_h^* 0 \\ &= 0 \end{aligned}$$

- $h > k$, and $1 \leq j \leq k$. Substituting gives

$$\begin{aligned} u_h^* M v_j &= u_h^* \left(\sigma_j u_j \right) \\ &= \sigma_j u_h^* u_j \\ &= 0 \end{aligned}$$

Defining matrices U and V with columns made up of the $\{u_h\}_{h=1}^n$ and $\{v_j\}_{j=1}^m$ completes the proof. $\sharp$

If $M = M^* \succeq 0$, then there is a unique matrix S satisfying

- $S = S^*$
- $S \succeq 0$ (moreover, $S \succ 0 \Leftrightarrow M \succ 0$)
- $S^2 = M$

S is called the *Hermitian square-root of* M and denoted $M^{\frac{1}{2}}$.

Facts:

1. Calculating the Hermitian square root of M:
   (a) Do a Schur decomposition of M, so $M = Q\Lambda Q^*$.
   (b) Since $M = M^*$, Λ is diagonal and real.
   (c) Since $M \succeq 0$, the diagonal entries of Λ are non-negative, denote them as $\lambda_1, \lambda_2, \ldots, \lambda_n$.
   (d) Define
   $$S := Q \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix} Q^*$$
   (e) Note that $S = S^* \succeq 0$, and $S^2 = M$.

2. If $M = M^* \succ 0$, then M is invertible, and M^{-1} is Hermitian and positive definite. Hence it has a Hermitian square root. In fact
   $$\left(M^{-1}\right)^{\frac{1}{2}} = \left(M^{\frac{1}{2}}\right)^{-1}$$
   so write $M^{-\frac{1}{2}}$ without any confusion as to its meaning.

Fact: Given $M \in \mathcal{H}^{n\times n}$ and $L \in \mathbf{C}^{n\times n}$, with L invertible. Then

$$M \succ 0 \;\Leftrightarrow L^*ML \succ 0$$

Fact: Given $X \in \mathcal{H}^{n\times n}$, $Y \in \mathcal{H}^{m\times m}$,

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \succ 0 \;\Leftrightarrow\; X \succ 0 \text{ and } Y \succ 0$$

Fact: Given $X \in \mathcal{H}^{n\times n}$, $Z \in \mathbf{F}^{n\times m}$,

$$\begin{bmatrix} X & Z \\ Z^* & I_m \end{bmatrix} \succ 0 \;\Leftrightarrow\; X - ZZ^* \succ 0$$

Proof: Use $L := \begin{bmatrix} I_n & 0 \\ -Z^* & I_m \end{bmatrix}$.

This leads to what is typically called the "Schur complement" theorem.

Fact: Given $X \in \mathcal{H}^{n\times n}$, $Y \in \mathcal{H}^{m\times m}$, $Z \in \mathbf{C}^{n\times m}$,

$$\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \succ 0 \;\Leftrightarrow\; Y \succ 0, \text{and } X - ZY^{-1}Z^* \succ 0$$

Proof: Note that if $Y \succ 0$,

$$\begin{bmatrix} I_n & 0 \\ 0 & Y^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & Y^{-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} X & ZY^{-\frac{1}{2}} \\ Y^{-\frac{1}{2}}Z^* & I_m \end{bmatrix}$$

Lemma: *Suppose* $X_{11} \in \mathbf{F}^{n\times n}$, $Y_{11} \in \mathbf{F}^{n\times n}$, *with* $X_{11} = X_{11}^* \succ 0$, *and* $Y_{11} = Y_{11}^* \succ 0$. *Let* r *be a non-negative integer. Then there exist* $X_{12} \in \mathbf{F}^{n\times r}, X_{22} \in \mathbf{F}^{r\times r}$ *such that* $X_{22} = X_{22}^*$, *and*

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \succ 0 \quad , \quad \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Y_{11} & ? \\ ? & ? \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X_{11} & I_n \\ I_n & Y_{11} \end{bmatrix} \succeq 0 \qquad \text{and} \qquad \text{rank} \begin{bmatrix} X_{11} & I_n \\ I_n & Y_{11} \end{bmatrix} \leq n + r$$

These last two conditions are equivalent to $X_{11} \succeq Y_{11}^{-1}$ *and* $\text{rank}\,(X_{11} - Y_{11}^{-1}) \leq r$.

Proof: Apply Schur Complement and Matrix inversion Lemmas...

$\Leftarrow$ By assumption, there is a matrix $L \in \mathbf{F}^{n\times r}$ such that $X_{11} - Y_{11}^{-1} = LL^*$. Defining $X_{12} := L$, and $X_{22} := I_r$ and note that

$$\begin{bmatrix} X_{11} & L \\ L^* & I_r \end{bmatrix}^{-1} = \begin{bmatrix} (X_{11} - LL^*)^{-1} & -(X_{11} - LL^*)^{-1} L \\ -L^* (X_{11} - LL^*)^{-1} & L^* (X_{11} - LL^*)^{-1} L + I_r \end{bmatrix} = \begin{bmatrix} Y_{11} & ? \\ ? & ? \end{bmatrix}$$

$\Rightarrow$ Using the matrix inversion lemma (item 1), it must be that

$$Y_{11}^{-1} = X_{11} - X_{12} X_{22}^{-1} X_{12}^*.$$

Hence, $X_{11} - Y_{11}^{-1} = X_{12} X_{22}^{-1} X_{12}^* \succeq 0$, and indeed,

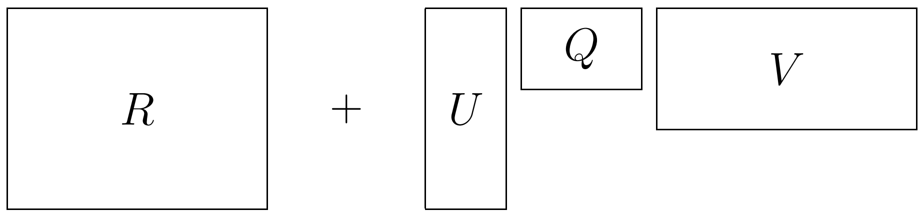
$$\text{rank}\left(X_{11} - Y_{11}^{-1}\right) = \text{rank}\left(X_{12} X_{22}^{-1} X_{12}^*\right) \leq r.$$

The other rank condition follows because

$$\begin{bmatrix} I_n & -Y_{11}^{-1} \\ 0 & I_n \end{bmatrix} \begin{bmatrix} X_{11} & I_n \\ I_n & Y_{11} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -Y_{11}^{-1} & I_n \end{bmatrix} = \begin{bmatrix} X_{11} - Y_{11}^{-1} & 0 \\ 0 & Y_{11} \end{bmatrix}$$

Lots of the control design algorithms we will study ($\mathcal{H}_\infty$, for instance) hinge on the following result from linear algebra:

1. Given $R \in \mathbf{F}^{l \times l}, U \in \mathbf{F}^{l \times m}$ and $V \in \mathbf{F}^{p \times l}$, where $m, p \leq l$.
2. We want to minimize $\bar{\sigma}\left[R + UQV\right]$ over $Q \in \mathbf{F}^{m \times p}$.



3. Suppose $U_\perp \in \mathbf{F}^{l \times (l-m)}$ and $V_\perp \in \mathbf{F}^{(l-p) \times l}$ have
   - $\begin{bmatrix} U & U_\perp \end{bmatrix}, \begin{bmatrix} V \\ V_\perp \end{bmatrix}$ are both invertible
   - $U^* U_\perp = 0_{m \times (l-m)}, V V_\perp^* = 0_{p \times (l-p)}$

Then

$$\inf_{Q \in \mathbf{F}^{m \times p}} \bar{\sigma}\left[R + UQV\right] < 1$$

if and only if

$$\begin{aligned} V_\perp \left(R^* R - I\right) V_\perp^* &\prec 0 \\ U_\perp^* \left(R R^* - I\right) U_\perp &\prec 0 \end{aligned}$$

Remark: Essentially, R must be smaller than 1 on the directions that U and V are perpendicular to.

Matrix dilation problems are of the form:

Given a partially specified matrix - when can the unspecified elements be chosen so that the full matrix has some property?

Already seen one type of problem. Next, we derive a main elementary matrix dilation theorem. We start simple and build...

Given $A \in \mathbf{C}^{m \times n}$, it is clear that

$$\min_{X \in \mathbf{C}^{q \times n}} \bar{\sigma} \begin{bmatrix} X \\ A \end{bmatrix} = \bar{\sigma}(A)$$

and this can easily be achieved by choosing $X := 0$. Pick some $\gamma > \bar{\sigma}(A)$. Characterize all X that give $\bar{\sigma} \begin{bmatrix} X \\ A \end{bmatrix} < \gamma$.

Lemma: Suppose $Y \in \mathbf{F}^{n \times n}$ is invertible. Then

$$\left\{ X \in \mathbf{F}^{q \times n} : X^*X \prec Y^*Y \right\} = \left\{ WY : W \in \mathbf{F}^{q \times n}, \bar{\sigma}(W) < 1 \right\}$$

Proof:

A simple chain of equivalences

$$\begin{aligned} X^*X \prec Y^*Y \;&\Leftrightarrow\; X^*X - Y^*Y \prec 0 \\ &\Leftrightarrow\; Y^{-*}\left[X^*X - Y^*Y\right]Y^{-1} \prec 0 \\ &\Leftrightarrow\; Y^{-*}X^*XY^{-1} - I \prec 0 \\ &\Leftrightarrow\; \bar{\sigma}\left(XY^{-1}\right) < 1 \\ &\Leftrightarrow\; \bar{\sigma}(W) < 1 \text{ and } W = XY^{-1} \\ &\Leftrightarrow\; \bar{\sigma}(W) < 1 \text{ and } X = WY \end{aligned}$$

The lemma easily gives

Lemma: Given $A \in \mathbf{F}^{m \times n}$, and $\gamma > \bar{\sigma}(A)$. Then

$$\left\{ X \in \mathbf{F}^{q \times n} : \bar{\sigma} \begin{bmatrix} X \\ A \end{bmatrix} < \gamma \right\} = \left\{ W \left(\gamma^2 I_n - A^* A\right)^{\frac{1}{2}} : W \in \mathbf{F}^{q \times n}, \bar{\sigma}(W) < 1 \right\}$$

Proof:

Another chain of equivalences

$$\begin{aligned} \bar{\sigma}\left(\begin{bmatrix} X \\ A \end{bmatrix} \right) < \gamma \;&\Leftrightarrow\; X^* X + A^* A - \gamma^2 I \prec 0 \\ &\Leftrightarrow\; X^* X \prec \gamma^2 I - A^* A \\ &\Leftrightarrow\; X^* X \prec \left(\gamma^2 I - A^* A\right)^{1/2} \left(\gamma^2 I - A^* A\right)^{1/2} \end{aligned}$$

Now apply previous Lemma.

Equivalently, for any $X \in \mathbf{F}^{q \times n}$ and $\gamma > \bar{\sigma}(A)$, we have

$$\bar{\sigma} \begin{bmatrix} X \\ A \end{bmatrix} < \gamma \quad \Leftrightarrow \quad \bar{\sigma}\left[X \left(\gamma^2 I_n - A^* A\right)^{-\frac{1}{2}} \right] < 1$$

Similarly, for $B \in \mathbf{F}^{q \times p}$, and $\gamma > \bar{\sigma}(B)$, we have

$$\left\{ X \in \mathbf{F}^{q \times n} : \bar{\sigma} \left[\begin{array}{cc} X & B \end{array} \right] < \gamma \right\} =$$

$$\left\{ \left(\gamma^2 I_q - BB^* \right)^{\frac{1}{2}} W : W \in \mathbf{F}^{q \times n}, \bar{\sigma}(W) < 1 \right\}$$

Along these lines, a corollary follows:

Corollary RV: Given $R \in \mathbf{F}^{n \times n}, V \in \mathbf{F}^{t \times n}$, with V full row rank. Then

$$\min_{Q \in \mathbf{F}^{n \times t}} \bar{\sigma}(R + QV) = \bar{\sigma}(RV_\perp^*)$$

where $V_\perp \in \mathbf{F}^{(n-t) \times n}$ satisfies

$$V_\perp V_\perp^* = I_{n-t} \quad , \quad V_\perp V^* = 0 \quad , \quad \det \left[\begin{array}{c} V \\ V_\perp \end{array} \right] \neq 0$$

Proof: let $S \in \mathbf{F}^{t\times t}$ be invertible such that $V_o := SV \in \mathbf{F}^{t\times n}$ satisfies $V_o V_o^* = I_t$. Then, for any $Q \in \mathbf{F}^{n\times t}$, we have

$$\begin{aligned} R + QV &= R + QS^{-1}SV \\ &= R + QS^{-1}V_o \end{aligned}$$

Since S is invertible, by picking Q, we equivalently have complete freedom in picking $Q_o (:= QS^{-1})$. Hence

$$\min_{Q\in\mathbf{F}^{n\times t}} \bar{\sigma}\left(R + QV\right) = \min_{Q_o\in\mathbf{F}^{n\times t}} \bar{\sigma}\left(R + Q_oV_o\right) =$$

Also,

$$T := \begin{bmatrix} V_o \\ V_\perp \end{bmatrix}$$

is a square, unitary matrix. Hence,

$$\min_{Q_o\in\mathbf{F}^{n\times t}} \bar{\sigma}\left(R + Q_oV_o\right) = \min_{Q_o\in\mathbf{F}^{n\times t}} \bar{\sigma}\left(\left(R + Q_oV_o\right)T^*\right)$$

But $\left(R + Q_oV_o\right)T^*$ is simply

$$\left(R + Q_oV_o\right)T^* = \begin{bmatrix} RV_o^* + Q_o & RV_\perp^* \end{bmatrix}$$

The minimum (over Q_o) that the maximum singular value can take on is clearly $\bar{\sigma}\left(RV_\perp^*\right)$, which is achieved when

$$Q_o := -RV_o^* = -RV^*S^*$$

and hence

$$\begin{aligned} Q &= Q_oS \\ &= -RV^*S^*S \\ &= -RV^*\left(VV^*\right)^{-1} \end{aligned}$$

Given $A \in \mathbf{F}^{m\times n}, B \in \mathbf{F}^{q\times p}, C \in \mathbf{F}^{m\times p}$, what is

$$\min_{X\in\mathbf{F}^{q\times n}} \bar{\sigma} \begin{bmatrix} X & B \\ A & C \end{bmatrix}$$

The theorem, independently (and in many different forms) by Sarason, Adamjan-Arov-Krien, Sz Nagy-Foias, Davis-Kahan-Weinberger, and Parrot is:

Theorem: Given A, B and C as above. Then

$$\min_{X\in\mathbf{F}^{q\times n}} \bar{\sigma} \begin{bmatrix} X & B \\ A & C \end{bmatrix} = \max\left\{ \bar{\sigma} \begin{bmatrix} A & C \end{bmatrix} \; , \; \bar{\sigma} \begin{bmatrix} B \\ C \end{bmatrix} \right\}$$

Remark: $X = 0$ typically does <u>not</u> achieve the minimum cost. Try a simple, real 2×2 example...

Note that the 2×2 block matrix can be written as

$$\begin{bmatrix} X & B \\ A & C \end{bmatrix} = \begin{bmatrix} 0 & B \\ A & C \end{bmatrix} + \begin{bmatrix} I_q \\ 0 \end{bmatrix} X \begin{bmatrix} I_n & 0 \end{bmatrix}$$

which is a special form of the $R + UQV$ expression.

Theorem: Given $A \in \mathbf{F}^{m \times n}, B \in \mathbf{F}^{q \times p}, C \in \mathbf{F}^{m \times p}$. Then

$$\min_{X \in \mathbf{F}^{q \times n}} \bar{\sigma} \begin{bmatrix} X & B \\ A & C \end{bmatrix} = \max \left\{ \bar{\sigma} \begin{bmatrix} A & C \end{bmatrix} \, , \, \bar{\sigma} \begin{bmatrix} B \\ C \end{bmatrix} \right\}$$

Proof: Clearly, nothing smaller than the right-hand-side is achievable. Take any $\gamma > \bar{\sigma} \begin{bmatrix} A & C \end{bmatrix}$. Then

$$\min_X \bar{\sigma} \begin{bmatrix} X & B \\ A & C \end{bmatrix} < \gamma \iff \min_X \bar{\sigma} \left(\begin{bmatrix} X & B \end{bmatrix} S^{-\frac{1}{2}} \right) < 1$$

where

$$S := \gamma^2 I - \begin{bmatrix} A^* \\ C^* \end{bmatrix} \begin{bmatrix} A & C \end{bmatrix}$$

Hence there exists an X such that $\bar{\sigma} \begin{bmatrix} X & B \\ A & C \end{bmatrix} < \gamma$ if and only if

$$\min_X \bar{\sigma} \left[\underbrace{X}_{Q} \underbrace{\begin{bmatrix} I & 0 \end{bmatrix} S^{-\frac{1}{2}}}_{V} + \underbrace{\begin{bmatrix} 0 & B \end{bmatrix} S^{-\frac{1}{2}}}_{R} \right] < 1$$

What should $V_\perp$ be? It needs to satisfy $V_\perp V^* = 0$ and $V_\perp V_\perp^* = I$. The first condition implies that

$$V_\perp V^* = 0 \iff V_\perp S^{-\frac{1}{2}} \begin{bmatrix} I \\ 0 \end{bmatrix} = 0$$

so that $V_\perp$ is of the form $V_\perp = \begin{bmatrix} 0 & L \end{bmatrix} S^{\frac{1}{2}}$ for some (at this point) arbitrary L. The second condition requires

$$V_\perp V_\perp^* = I \implies L \left(\gamma^2 I - C^* C \right) L^* = I$$

so that $L = \left(\gamma^2 I - C^* C \right)^{-\frac{1}{2}}$ is a suitable choice.

Hence, the original equivalence continues,

$$\begin{aligned} \min_X \bar{\sigma}\left(QV + R\right) < 1 &\iff \bar{\sigma}\left(RV_\perp\right) < 1 \\ &\iff \bar{\sigma}\left[B\left(\gamma^2 I - C^*C\right)^{-\frac{1}{2}}\right] < 1 \\ &\iff \bar{\sigma}\begin{bmatrix} B \\ C \end{bmatrix} < \gamma \end{aligned}$$

Hence, any γ larger than both $\bar{\sigma}\left[A\ C\right]$ and $\bar{\sigma}\begin{bmatrix} B \\ C \end{bmatrix}$ is achievable, using, for instance

$$X := -B\left(\gamma^2 I - C^*C\right)^{-1} C^*A$$

Moreover (though we do not explicitly use it) the minimum is achieved (compactness argument).

Partial answer to the $R + UQV$ problem when similarity scalings are included:

1. Let $R, U, V, U_\perp$ and $V_\perp$ be given as before.
2. Let $\mathcal{Z} \subset \mathbf{F}^{l \times l}$ be a given set of positive definite, Hermitian matrices

Then

$$\inf_{\substack{Q \in \mathbf{F}^{m \times p} \\ Z \in \mathcal{Z}}} \bar{\sigma} \left[Z^{1/2} \left(R + UQV \right) Z^{-1/2} \right] < 1$$

if and only if there is a $Z \in \mathcal{Z}$ such that

$$V_\perp \left(R^* Z R - Z \right) V_\perp^* \prec 0$$

and

$$U_\perp^* \left(R Z^{-1} R^* - Z^{-1} \right) U_\perp \prec 0.$$

Proof: For each fixed $Z \in \mathcal{Z}$, consider the problem

$$\beta(Z) := \inf_{Q \in \mathbf{F}^{r \times t}} \bar{\sigma}\left[Z^{\frac{1}{2}}(R + UQV)Z^{-\frac{1}{2}}\right]$$

Define $\tilde{R} := Z^{\frac{1}{2}} R Z^{-\frac{1}{2}}, \tilde{U} := Z^{\frac{1}{2}} U, \tilde{V} = V Z^{-\frac{1}{2}}$. Note that the columns of of $Z^{-\frac{1}{2}} U_\perp$ span the space orthogonal to the range (column) of $\tilde{U}$, since $\left(Z^{-\frac{1}{2}} U_\perp\right)^* \tilde{U} = 0$. Similarly, the rows of $V_\perp Z^{\frac{1}{2}}$ span the space orthogonal to the range (row) of $\tilde{V}$. Therefore, for fixed $Z \in \mathcal{Z}$, $\beta(Z) < \alpha$ if and only if

$$U_\perp^* Z^{-\frac{1}{2}}\left(Z^{\frac{1}{2}} R Z^{-\frac{1}{2}} Z^{-\frac{1}{2}} R^* Z^{\frac{1}{2}} - \alpha^2 I\right) Z^{-\frac{1}{2}} U_\perp \prec 0,$$

and

$$V_\perp Z^{\frac{1}{2}}\left(Z^{-\frac{1}{2}} R^* Z^{\frac{1}{2}} Z^{\frac{1}{2}} R Z^{-\frac{1}{2}} - \alpha^2 I\right) Z^{\frac{1}{2}} V_\perp^* \prec 0.$$

These simplify to

$$U_\perp^*\left(R Z^{-1} R^* - \alpha^2 Z^{-1}\right) U_\perp \prec 0, \tag{1}$$

and

$$V_\perp\left(R^* Z R - \alpha^2 Z\right) V_\perp^* \prec 0 \tag{2}$$

as claimed. $\sharp$

The previous results are directly useful in discrete-time problems.

Using similar techniques, the analogous theorem for definiteness can be proven:

Theorem: Given $R \in \mathbf{F}^{l \times l}, U \in \mathbf{F}^{l \times m}$ and $V \in \mathbf{F}^{p \times l}$, where $m, p \leq l$. Suppose $U_\perp \in \mathbf{F}^{l \times (l-m)}$ and $V_\perp \in \mathbf{F}^{(l-p) \times l}$ have

- $\begin{bmatrix} U & U_\perp \end{bmatrix}, \begin{bmatrix} V \\ V_\perp \end{bmatrix}$ are both invertible
- $U^* U_\perp = 0_{m \times (l-m)}, V V_\perp^* = 0_{p \times (l-p)}$

Then, there exist a $Q \in \mathbf{F}^{m \times p}$ such that

$$[R + UQV] + [R + UQV]^* \prec 0$$

if and only if

$$U_\perp^* \left(R + R^*\right) U_\perp \prec 0, \quad V_\perp \left(R + R^*\right) V_\perp^* \prec 0$$

Lemma: $S = S^* \succ 0$, T given square matrices. For every K,

$$-TK^* - KT^* + KSK \succeq -TS^{-1}T^*.$$

Furthermore, $K_0 := TS^{-1}$ achieves equality.

Proof: Complete squares as

$$\begin{aligned}
&-TK^* - KT^* + KSK \\
&\quad = \left(KS^{1/2} - TS^{-1/2}\right)\left(KS^{1/2} - TS^{-1/2}\right)^* - TS^{-1}T^* \\
&\quad \succeq -TS^{-1}T^*
\end{aligned}$$

Note that equality is achieved by making $KS^{1/2} - TS^{-1/2} = 0$, which can be accomplished with $K = TS^{-1}$.

Lemma: $S = S^* \succeq 0$, $\mathrm{Ker}S \subseteq \mathrm{Ker}T$. Let K_0 be any solution of the equation $K_0 S = T$. Then for every K

$$-TK^* - KT^* + KSK \succeq -TK_0^* - K_0T^* + K_0SK_0 \, (= -K_0SK_0)$$

Proof: For any K,

$$\begin{aligned}
&T\left(K_0 - K\right)^* + \left(K_0 - K\right)T^* - K_0SK_0^* + KSK \\
&\quad = \left(K_0 - K\right)S\left(K_0 - K\right)^* \\
&\quad \succeq 0
\end{aligned}$$

To verify the equality, simply substitute for T. Also note that the equation $K_0 S = T$ may have many solutions. If $K_{0,1}$ and $K_{0,2}$ are two such solutions, then by making the argument twice above, we have

$$K_{0,1}SK_{0,1}^* = K_{0,2}SK_{0,2}^*$$

Equivalently, $TK_{0,1} = TK_{0,2}$.