Nonlinear Systems and Control
Lecture # 9
Lyapunov Stability

Quadratic Forms

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j, \quad P = P^T$$

$$\lambda_{\min}(P)\|x\|^2 \le x^T P x \le \lambda_{\max}(P)\|x\|^2$$

$P \ge 0$ (Positive semidefinite) if and only if $\lambda_i(P) \ge 0 \ \forall i$

$P > 0$ (Positive definite) if and only if $\lambda_i(P) > 0 \ \forall i$

$V(x)$ is positive definite if and only if P is positive definite

$V(x)$ is positive semidefinite if and only if P is positive semidefinite

$P > 0$ if and only if all the leading principal minors of P are positive

Linear Systems

$$\dot{x} = Ax$$

$$V(x) = x^T P x, \qquad P = P^T > 0$$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x \stackrel{\text{def}}{=} -x^T Q x$$

If $Q > 0$, then A is Hurwitz

Or choose $Q > 0$ and solve the Lyapunov equation

$$PA + A^T P = -Q$$

If $P > 0$, then A is Hurwitz

Matlab: $P =$ lyap(A', Q)

Theorem A matrix A is Hurwitz if and only if for any $Q = Q^T > 0$ there is $P = P^T > 0$ that satisfies the Lyapunov equation

$$PA + A^T P = -Q$$

Moreover, if A is Hurwitz, then P is the unique solution

Idea of the proof: Sufficiency follows from Lyapunov's theorem. Necessity is shown by verifying that

$$P = \int_0^\infty \exp(A^T t) Q \exp(At)\, dt$$

is positive definite and satisfies the Lyapunov equation

Linearization Consider $\dot{x} = f(x)$. $f(x)$ is continuously differentiable in D. By the mean value theorem:

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(z_i)x$$

with $z_i = \alpha_i x$, and $0 < \alpha_i < 1$.
Let

$$F(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x}(z_1) \\ \vdots \\ \frac{\partial f_n}{\partial x}(z_n) \end{pmatrix}.$$

Then $f(x) = F(x)x$ and $F(0) = \frac{\partial f}{\partial x}(0) = A$.

$$\dot{x} = f(x) = F(x)x = [A + G(x)]x$$

Linearization $\dot{x} = f(x) = F(x)x = [A + G(x)]x$. By continuity of $\frac{\partial f_i}{\partial x}$, we have

$$G(x) := F(x) - A \to 0 \text{ as } x \to 0$$

Suppose A is Hurwitz. Choose $Q = Q^T > 0$ and solve the Lyapunov equation $PA + A^T P = -Q$ for P. Use $V(x) = x^T P x$ as a Lyapunov function candidate for $\dot{x} = f(x)$

$$\begin{aligned}
\dot{V}(x) &= x^T P f(x) + f^T(x) P x \\
&= x^T P[A + G(x)]x + x^T[A^T + G^T(x)]Px \\
&= x^T(PA + A^T P)x + 2x^T PG(x)x \\
&= -x^T Q x + 2x^T PG(x)x
\end{aligned}$$

$$\dot{V}(x) \leq -x^T Q x + 2\|P\| \, \|G(x)\| \, \|x\|^2$$

For any $\gamma > 0$, there exists $r > 0$ such that

$$\|G(x)\| < \gamma, \;\; \forall \; \|x\| < r$$

$$x^T Q x \geq \lambda_{\min}(Q)\|x\|^2 \;\Leftrightarrow\; -x^T Q x \leq -\lambda_{\min}(Q)\|x\|^2$$

$$\dot{V}(x) < -[\lambda_{\min}(Q) - 2\gamma\|P\|]\|x\|^2, \;\; \forall \; \|x\| < r$$

Choose

$$\gamma < \frac{\lambda_{\min}(Q)}{2\|P\|}$$

$V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = f(x)$

We can use $V(x) = x^T P x$ to estimate the region of attraction

$$\text{Suppose} \quad \dot{V}(x) < 0, \quad \forall\, 0 < \|x\| < r$$

$$\text{Take} \quad c = \min_{\|x\|=r} x^T P x = \lambda_{\min}(P) r^2$$

$$\{x^T P x < c\} \subset \{\|x\| < r\}$$

All trajectories starting in the set $\{x^T P x < c\}$ approach the origin as t tends to ∞.
Hence, the set $\{x^T P x < c\}$ is a subset of the region of attraction (an estimate of the region of attraction)

Example

$$\begin{aligned}\dot{x}_1 &= -x_2\\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2\end{aligned}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

has eigenvalues $(-1 \pm j\sqrt{3})/2$. Hence the origin is asymptotically stable

Take $Q = I$, $\quad PA + A^T P = -I \;\Rightarrow\; P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

$\lambda_{\min}(P) = 0.691$

$$V(x) = x^T P x = 1.5x_1^2 - x_1x_2 + x_2^2$$

$$\begin{aligned}\dot{V}(x) &= (3x_1 - x_2)(-x_2) + (-x_1 + 2x_2)[x_1 + (x_1^2 - 1)x_2] \\ &= -(x_1^2 + x_2^2) - (x_1^3x_2 - 2x_1^2x_2^2)\end{aligned}$$

$$\dot{V}(x) \le -\|x\|^2 + |x_1|\,|x_1x_2|\,|x_1 - 2x_2| \le -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4$$

where $|x_1| \le \|x\|,\ |x_1x_2| \le \frac{1}{2}\|x\|^2,\ |x_1 - 2x_2| \le \sqrt{5}\|x\|$

$$\dot{V}(x) < 0 \text{ for } 0 < \|x\|^2 < \frac{2}{\sqrt{5}} \stackrel{\text{def}}{=} r^2$$

Take $c = \lambda_{\min}(P)r^2 = 0.691 \times \dfrac{2}{\sqrt{5}} = 0.618$

$\{V(x) < c\}$ is an estimate of the region of attraction

Example:

$$\dot{x} = -g(x)$$

$$g(0) = 0; \quad xg(x) > 0, \quad \forall\, x \neq 0 \ \text{ and } \ x \in (-a, a)$$

$$V(x) = \int_0^x g(y)\, dy$$

$$\dot{V}(x) = \frac{\partial V}{\partial x}[-g(x)] = -g^2(x) < 0, \, \forall\, x \in (-a, a), \ x \neq 0$$

The origin is asymptotically stable

If $xg(x) > 0$ for all $x \neq 0$, use

$$V(x) = \tfrac{1}{2}x^2 + \int_0^x g(y)\, dy$$

$$V(x) = \frac{1}{2}x^2 + \int_0^x g(y)\ dy$$

is positive definite for all x and radially unbounded since $V(x) \geq \frac{1}{2}x^2$

$$\dot{V}(x) = -xg(x) - g^2(x) < 0, \quad \forall\ x \neq 0$$

The origin is globally asymptotically stable

Example: Pendulum equation without friction

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\,a\sin x_1\end{aligned}$$

$$V(x) = a(1-\cos x_1) + \tfrac{1}{2}x_2^2$$

$V(0) = 0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$

$$\dot{V}(x) = a\dot{x}_1\sin x_1 + x_2\dot{x}_2 = ax_2\sin x_1 - ax_2\sin x_1 = 0$$

The origin is stable

Since $\dot{V}(x) \equiv 0$, the origin is not asymptotically stable

Example: Pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - bx_2\end{aligned}$$

$$V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -bx_2^2$$

The origin is stable

$\dot{V}(x)$ is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1

The conditions of Lyapunov's theorem are only sufficient. Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate

Try

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^T P x + a(1 - \cos x_1) \\ &= \tfrac{1}{2}[x_1 \ x_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1 - \cos x_1) \end{aligned}$$

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

$$
\begin{aligned}
\dot{V}(x) &= (p_{11}x_1 + p_{12}x_2 + a\sin x_1)\,x_2 \\
&\quad + (p_{12}x_1 + p_{22}x_2)\,(-a\sin x_1 - bx_2) \\
&= a(1 - p_{22})x_2 \sin x_1 - ap_{12}x_1 \sin x_1 \\
&\quad + (p_{11} - p_{12}b)\,x_1x_2 + (p_{12} - p_{22}b)\,x_2^2
\end{aligned}
$$

$p_{22} = 1, \quad p_{11} = bp_{12} \;\Rightarrow\; 0 < p_{12} < b,$ Take $p_{12} = b/2$

$$\dot{V}(x) = -\tfrac{1}{2}abx_1 \sin x_1 - \tfrac{1}{2}bx_2^2$$

$$D = \{x \in R^2 \mid |x_1| < \pi\}$$

$V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D
The origin is asymptotically stable

Read about the variable gradient method in the textbook