Nonlinear Systems and Control
Lecture # 8
Lyapunov Stability

Let $V(x)$ be a continuously differentiable function defined in a domain $D \subset R^n$; $0 \in D$. The derivative of V along the trajectories of $\dot{x} = f(x)$ is

$$
\begin{aligned}
\dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\
&= \left[\frac{\partial V}{\partial x_1}, \ \frac{\partial V}{\partial x_2}, \ \cdots, \ \frac{\partial V}{\partial x_n} \right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \\
&= \frac{\partial V}{\partial x} f(x) =: L_f V(x)
\end{aligned}
$$

It is the *Lie Derivative* of V with respect to f or along f

If $\phi(t;x)$ is the solution of $\dot{x} = f(x)$ that starts at initial state x at time $t = 0$, then

$$\dot{V}(x) = \frac{d}{dt} V(\phi(t;x))\Big|_{t=0}$$

If $\dot{V}(x)$ is negative, V will decrease along the solution of $\dot{x} = f(x)$

If $\dot{V}(x)$ is positive, V will increase along the solution of $\dot{x} = f(x)$

Lyapunov's Theorem:

- If there is $V(x)$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall\, x \in D \setminus \{0\}$$

$$\dot{V}(x) \leq 0, \quad \forall\, x \in D$$

then the origin is a stable

- Moreover, if

$$\dot{V}(x) < 0, \quad \forall\, x \in D \setminus \{0\}$$

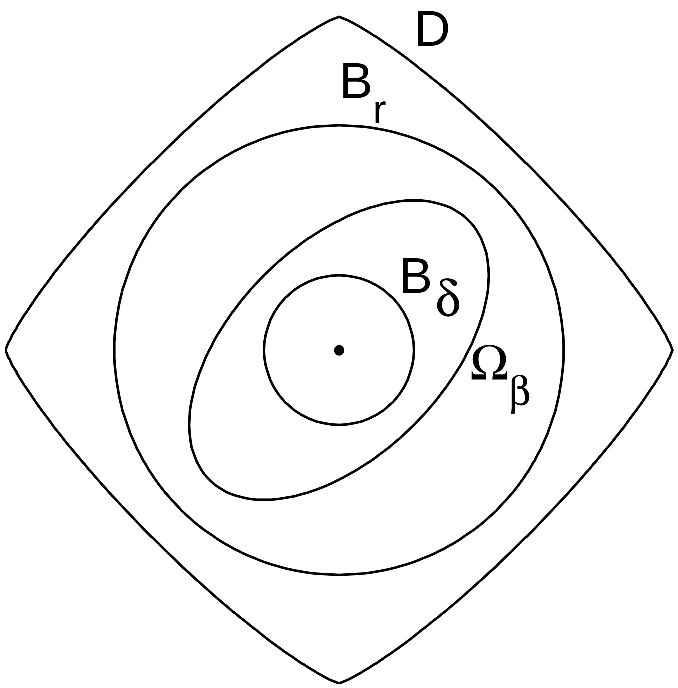
then the origin is asymptotically stable

- Furthermore, if $V(x) > 0$, $\forall\, x \neq 0$,

$$\|x\| \to \infty \;\Rightarrow\; V(x) \to \infty$$

and $\dot{V}(x) < 0$, $\forall\, x \neq 0$, then the origin is globally asymptotically stable

Proof:



$$0 < r \le \varepsilon, \;\; B_r = \{\|x\| \le r\}$$

$$\alpha = \min_{\|x\|=r} V(x) > 0$$

$$0 < \beta < \alpha$$

$$\Omega_\beta = \{x \in B_r \mid V(x) \le \beta\}$$

$$\|x\| \le \delta \;\Rightarrow\; V(x) < \beta$$

$$\|x\| \leq \delta \;\Rightarrow\; V(x) < \beta, \text{ i.e., } B_\delta \subset \Omega_\beta$$

Why?

Since $V(x)$ is continuous and $V(0) = 0$, $\forall \beta$, $\exists \delta'$ such that

$$\|x - 0\| < \delta' \Rightarrow \|V(x) - V(0)\| < \beta$$

Then Ω_β is in the interior of B_r. Why?

(Proof by contradiction): Suppose not, i.e., Ω_β is not in the interior of B_r. $\exists$ a point $p \in \Omega_\beta$ that lies on ∂B_r. At p, $V(p) \geq \alpha = \min_{\|x\|=r} V(x) > \beta$ (by definition of β), which is a contradiction ($\forall x \in \Omega_\beta, V(x) \leq \beta$).

Solutions starting in Ω_β stay in Ω_β because $\dot{V}(x) \le 0$ in Ω_β

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \le \varepsilon, \quad \forall\, t \ge 0$$

$\Rightarrow$ The origin is stable

Now suppose $\dot{V}(x) < 0 \;\forall\, x \in D \setminus \{0\}$.
$V(x(t))$ is monotonically decreasing and $V(x(t)) \ge 0$

$$\lim_{t\to\infty} V(x(t)) = c \ge 0$$

$$\lim_{t\to\infty} V(x(t)) = c \ge 0 \quad \text{Show that } c = 0$$

Suppose $c > 0$. By continuity of $V(x)$, there is $d > 0$ such that $B_d \subset \Omega_c$. Then, $x(t)$ lies outside B_d for all $t \ge 0$

$$\gamma = - \max_{d \le \|x\| \le r} \dot{V}(x)$$

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \le V(x(0)) - \gamma t$$

This inequality contradicts the assumption $c > 0$

$\Rightarrow$ The origin is asymptotically stable

The condition $\|x\| \to \infty \;\Rightarrow\; V(x) \to \infty$ implies that the set $\Omega_c = \{x \in R^n \mid V(x) \le c\}$ is compact for every $c > 0$. This is so because for any $c > 0$, there is $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Thus, $\Omega_c \subset B_r$.
All solutions starting Ω_c will converge to the origin. For any point $p \in R^n$, choosing $c = V(p)$ ensures that $p \in \Omega_c$

$\Rightarrow$ The origin is globally asymptotically stable

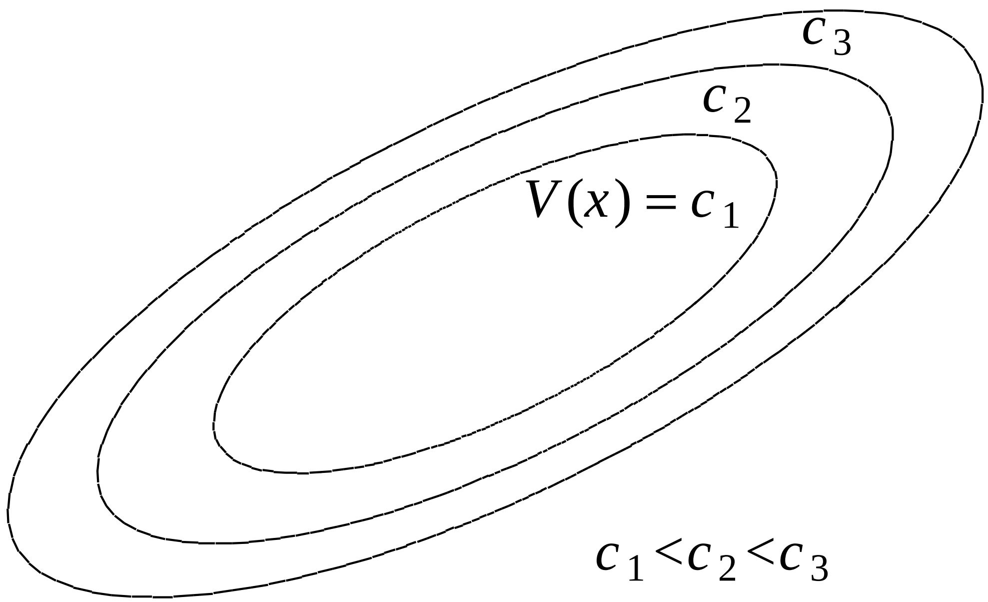
Terminology

$V(0) = 0,\ V(x) \geq 0$ for $x \neq 0$	Positive semidefinite
$V(0) = 0,\ V(x) > 0$ for $x \neq 0$	Positive definite
$V(0) = 0,\ V(x) \leq 0$ for $x \neq 0$	Negative semidefinite
$V(0) = 0,\ V(x) < 0$ for $x \neq 0$	Negative definite
$\|x\| \to \infty \ \Rightarrow \ V(x) \to \infty$	Radially unbounded

Lyapunov' Theorem: The origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite. It is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded

A continuously differentiable function $V(x)$ satisfying the conditions for stability is called a *Lyapunov function*.

The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov surface* or a *level surface*



Why do we need the radial unboundedness condition to show global asymptotic stability?

It ensures that $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is bounded for every $c > 0$

Without it Ω_c might not bounded for large c

Example

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$