

Nonlinear Systems and Control

Lecture # 5

Limit Cycles

Oscillation: A system oscillates when it has a **nontrivial periodic solution**

$$x(t+T) = x(t), \quad \forall t \geq 0$$

Linear (Harmonic) Oscillator:

$$\dot{z} = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} z$$

$$z_1(t) = r_0 \cos(\beta t + \theta_0), \quad z_2(t) = r_0 \sin(\beta t + \theta_0)$$

$$r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \quad \theta_0 = \tan^{-1} \left[\frac{z_2(0)}{z_1(0)} \right]$$

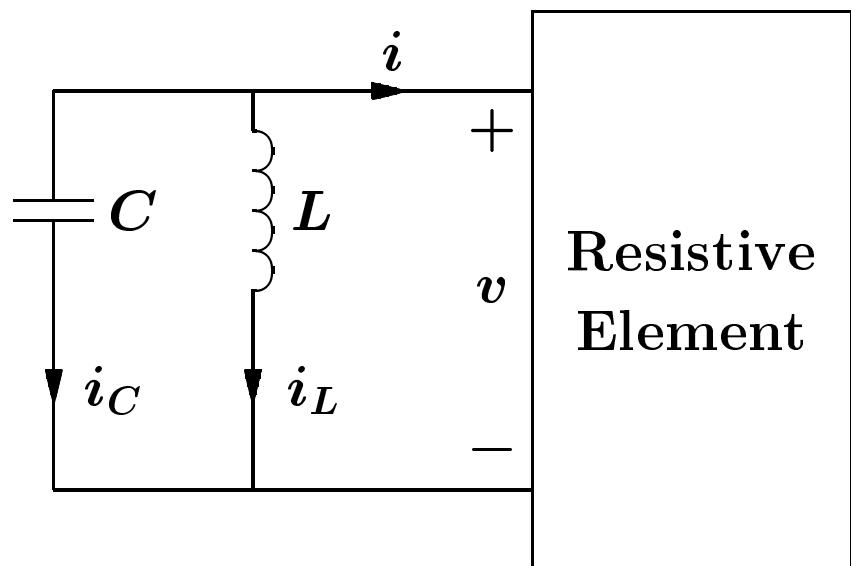
The linear oscillation is not practical because

- It is not structurally stable. Infinitesimally small perturbations may change the type of the equilibrium point to a stable focus (decaying oscillation) or unstable focus (growing oscillation)
- The amplitude of oscillation depends on the initial conditions

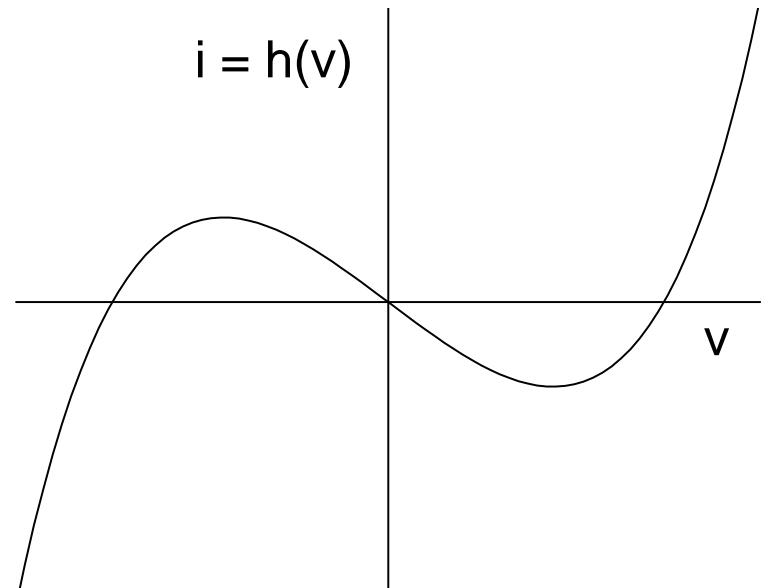
The same problems exist with oscillation of nonlinear systems due to a center equilibrium point (e.g., pendulum without friction)

Limit Cycles:

Example: Negative Resistance Oscillator



(a)



(b)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - \varepsilon h'(x_1)x_2$$

There is a unique equilibrium point at the origin

$$A = \frac{\partial f}{\partial x} \Big|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}$$

$$\lambda^2 + \varepsilon h'(0)\lambda + 1 = 0$$

$h'(0) < 0 \Rightarrow$ Unstable Focus or Unstable Node

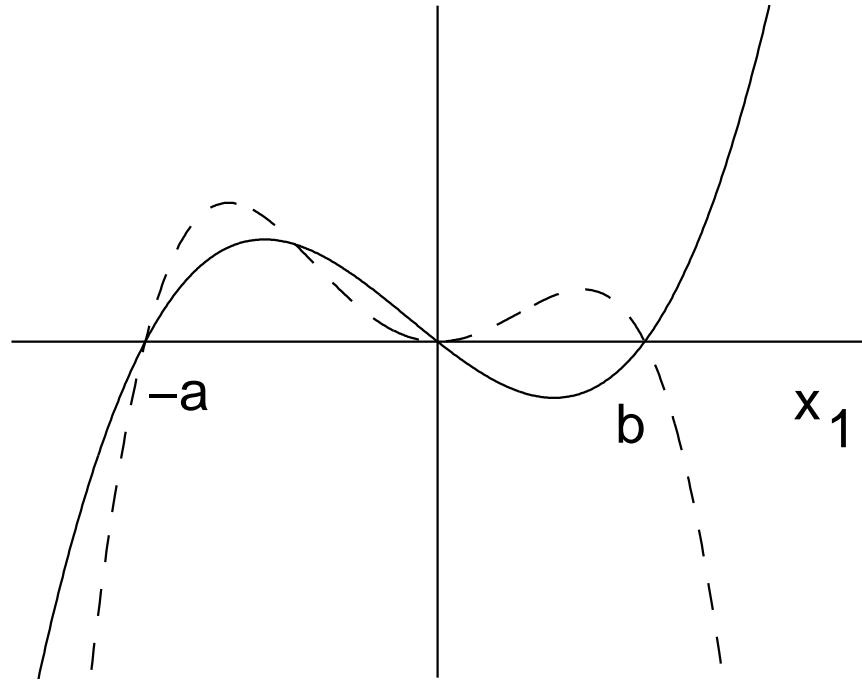
Energy Analysis:

$$E = \frac{1}{2}Cv_C^2 + \frac{1}{2}Li_L^2$$

$$v_C = x_1 \quad \text{and} \quad i_L = -h(x_1) - \frac{1}{\varepsilon}x_2$$

$$E = \frac{1}{2}C\{x_1^2 + [\varepsilon h(x_1) + x_2]^2\}$$

$$\begin{aligned}\dot{E} &= C\{x_1\dot{x}_1 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)\dot{x}_1 + \dot{x}_2]\} \\ &= C\{x_1x_2 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)x_2 - x_1 - \varepsilon h'(x_1)x_2]\} \\ &= C[x_1x_2 - \varepsilon x_1 h(x_1) - x_1x_2] \\ &= -\varepsilon C x_1 h(x_1)\end{aligned}$$

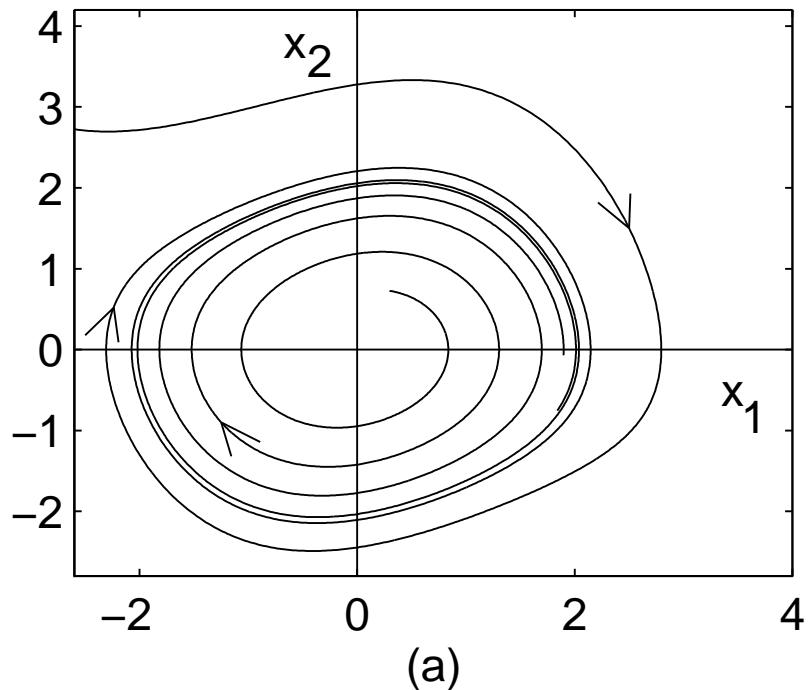


$$\dot{E} = -\varepsilon C x_1 h(x_1)$$

Example: Van der Pol Oscillator

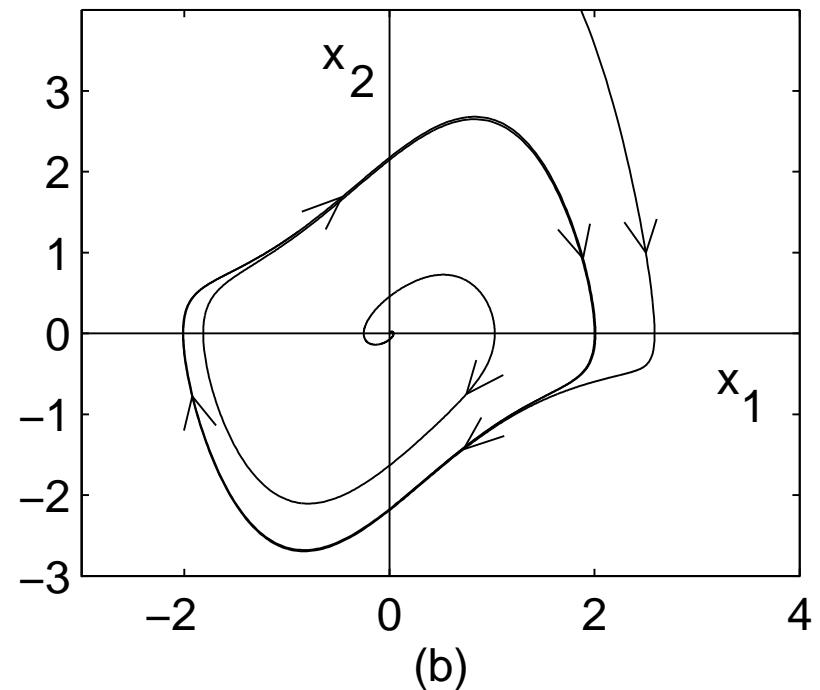
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2$$



(a)

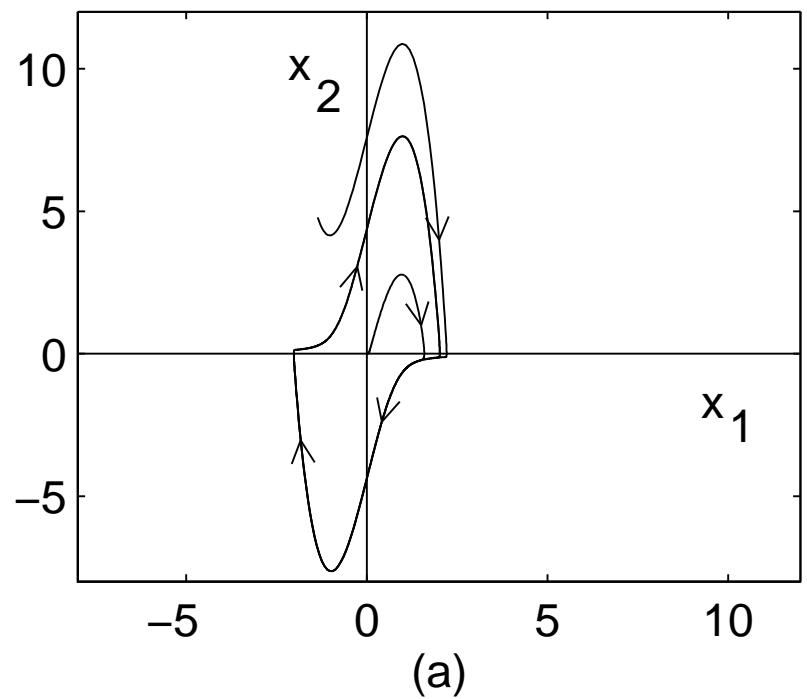
$$\varepsilon = 0.2$$



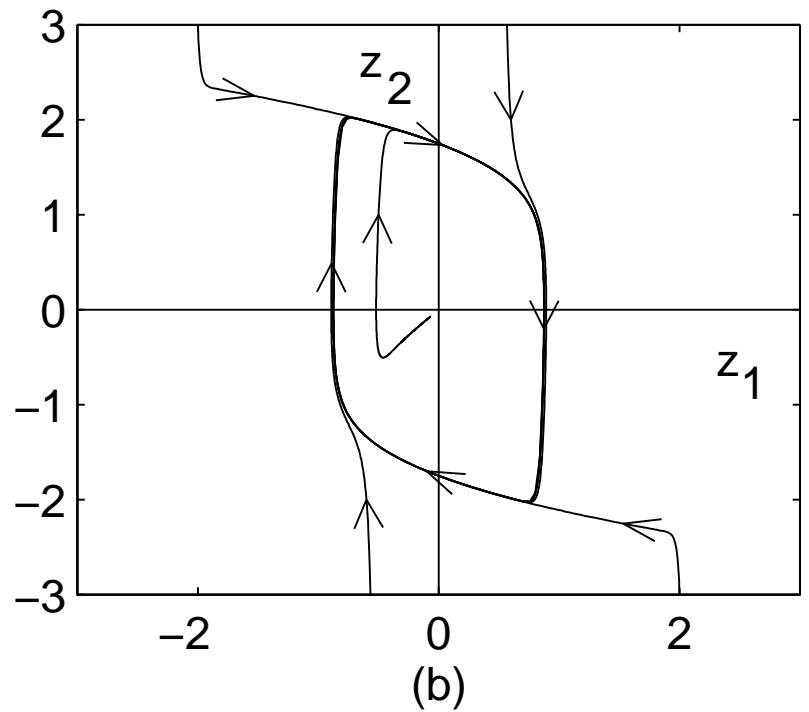
(b)

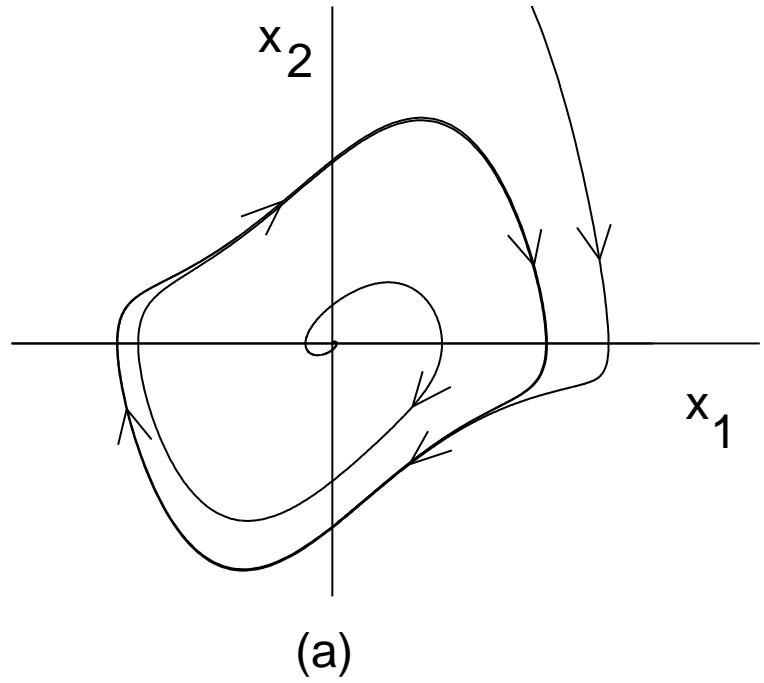
$$\varepsilon = 1$$

$$\begin{aligned}\dot{z}_1 &= \frac{1}{\varepsilon} z_2 \\ \dot{z}_2 &= -\varepsilon(z_1 - z_2 + \frac{1}{3}z_2^3)\end{aligned}$$



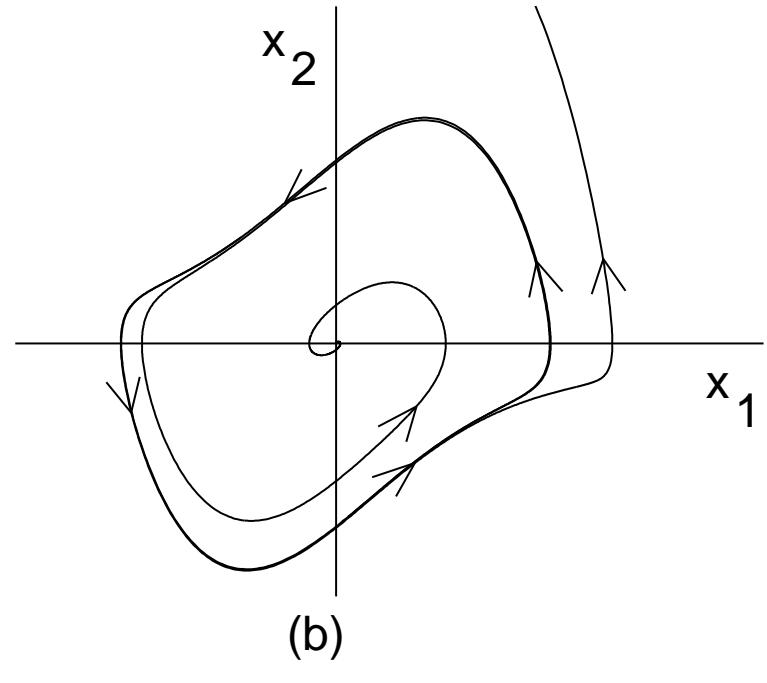
$$\varepsilon = 5$$





(a)

Stable Limit Cycle



(b)

Unstable Limit Cycle