Nonlinear Systems and Control
Lecture # 4
Qualitative Behavior Near Equilibrium Points
&
Multiple Equilibria

The qualitative behavior of a nonlinear system near an equilibrium point can take one of the patterns we have seen with linear systems. Correspondingly the equilibrium points are classified as stable node, unstable node, saddle, stable focus, unstable focus, or center

Can we determine the type of the equilibrium point of a nonlinear system by linearization?

Let $p = (p_1, p_2)$ be an equilibrium point of the system

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2)$$

where f_1 and f_2 are continuously differentiable
Expand f_1 and f_2 in Taylor series about (p_1, p_2)

$$\begin{aligned} \dot{x}_1 &= f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + \text{H.O.T.} \\ \dot{x}_2 &= f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + \text{H.O.T.} \end{aligned}$$

$$a_{11} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{12} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

$$a_{21} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{22} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

$$f_1(p_1, p_2) = f_2(p_1, p_2) = 0$$

$$y_1 = x_1 - p_1 \qquad y_2 = x_2 - p_2$$

$$\begin{aligned}
\dot{y}_1 &= \dot{x}_1 = a_{11}y_1 + a_{12}y_2 + \text{H.O.T.} \\
\dot{y}_2 &= \dot{x}_2 = a_{21}y_1 + a_{22}y_2 + \text{H.O.T.}
\end{aligned}$$

$$\dot{y} \approx Ay$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ & \\ a_{21} & a_{22} \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ & \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=p} = \left.\frac{\partial f}{\partial x}\right|_{x=p}$$

Eigenvalues of $\boldsymbol{A}$	Type of equilibrium point of the nonlinear system
$\boldsymbol{\lambda_2 < \lambda_1 < 0}$	Stable Node
$\boldsymbol{\lambda_2 > \lambda_1 > 0}$	Unstable Node
$\boldsymbol{\lambda_2 < 0 < \lambda_1}$	Saddle
$\boldsymbol{\alpha \pm j\beta,\ \alpha < 0}$	Stable Focus
$\boldsymbol{\alpha \pm j\beta,\ \alpha > 0}$	Unstable Focus
$\boldsymbol{\pm j\beta}$	Linearization Fails

Example

$$\begin{aligned} \dot{x}_1 &= -x_2 - \mu x_1 (x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 - \mu x_2 (x_1^2 + x_2^2) \end{aligned}$$

$x = 0$ is an equilibrium point

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\mu(3x_1^2 + x_2^2) & -(1 + 2\mu x_1 x_2) \\ (1 - 2\mu x_1 x_2) & -\mu(x_1^2 + 3x_2^2) \end{bmatrix}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$x_1 = r\cos\theta \text{ and } x_2 = r\sin\theta \;\Rightarrow\; \dot{r} = -\mu r^3 \text{ and } \dot{\theta} = 1$$

Stable focus when $\mu > 0$ and Unstable focus when $\mu < 0$

For a saddle point, we can use linearization to generate the stable and unstable trajectories

Let the eigenvalues of the linearization be $\boldsymbol{\lambda_1 > 0 > \lambda_2}$ and the corresponding eigenvectors be $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$

The stable and unstable trajectories will be tangent to the stable and unstable eigenvectors, respectively, as they approach the equilibrium point $\boldsymbol{p}$

For the unstable trajectories use $\boldsymbol{x_0 = p \pm \alpha v_1}$

For the stable trajectories use $\boldsymbol{x_0 = p \pm \alpha v_2}$

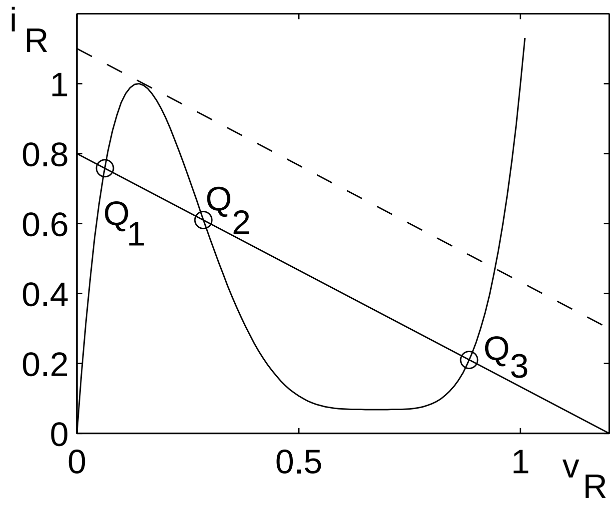
$\boldsymbol{\alpha}$ is a small positive number

Multiple Equilibria

Example: Tunnel-diode circuit

$$\begin{aligned}\dot{x}_1 &= 0.5[-h(x_1) + x_2] \\ \dot{x}_2 &= 0.2(-x_1 - 1.5x_2 + 1.2)\end{aligned}$$

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$$



$$\begin{aligned}Q_1 &= (0.063, 0.758) \\ Q_2 &= (0.285, 0.61) \\ Q_3 &= (0.884, 0.21)\end{aligned}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h'(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

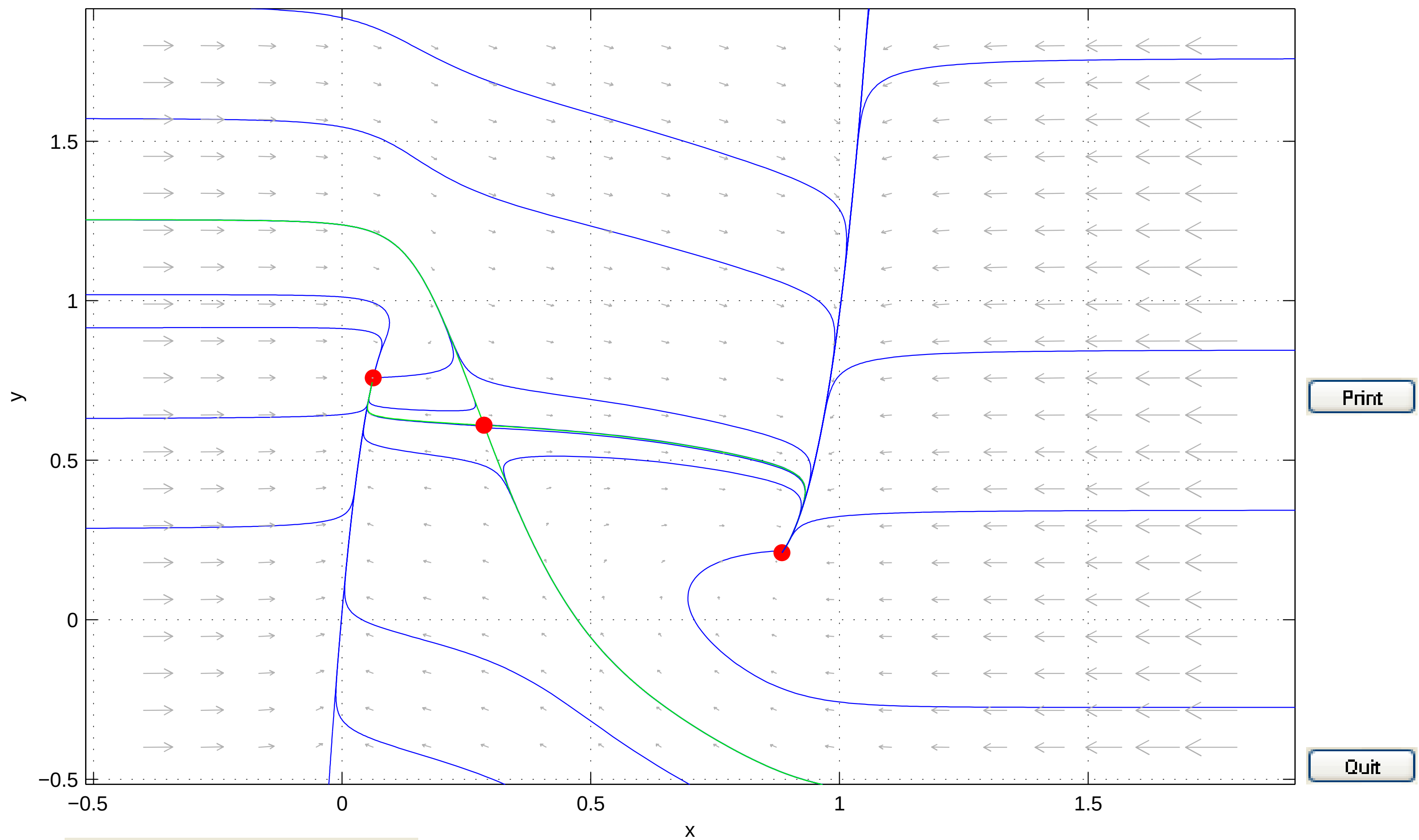
$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues} : -3.57, \ -0.33$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues} : 1.77, \ -0.25$$

$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues} : -1.33, \ -0.4$$

Q_1 is a stable node; Q_2 is a saddle; Q_3 is a stable node

$x' = 0.5\,(-17.76\,x + 103.79\,x^2 - 229.62\,x^3 + 226.31\,x^4 - 83.72\,x^5 + y)$
$y' = 0.2\,(-x - 1.5\,y + 1.2)$



Cursor position: (1.02, -0.908)

The second unstable trajectory --> a possible eq. pt. near (0.063, 0.76).
Ready.
The forward orbit from (1.7, 2.2) --> a possible eq. pt. near (0.88, 0.21).
The backward orbit from (1.7, 2.2) left the computation window.
Ready.

Hysteresis characteristics of the tunnel-diode circuit

$$u = E, \quad y = v_R$$