Nonlinear Systems and Control
Lecture # 37

Observers

Linearization
and
Extended Kalman Filter (EKF)

Linear Observer via Linearization

$$\dot{x} = f(x, u), \qquad y = h(x)$$

$$0 = f(x_{ss}, u_{ss}), \qquad y_{ss} = h(x_{ss})$$

Linearize about the equilibrium point:

$$\dot{x}_\delta = Ax_\delta + Bu_\delta, \qquad y_\delta = Cx_\delta$$

$$x_\delta = x - x_{ss}, \qquad u_\delta = u - u_{ss}, \qquad y_\delta = y - y_{ss}$$

What are A, B, C?

$$\dot{\hat{x}}_\delta = A\hat{x}_\delta + Bu_\delta + H(y_\delta - C\hat{x}_\delta), \qquad \hat{x} = x_{ss} + \hat{x}_\delta$$

$$(A - HC) \text{ is Hurwitz}$$

It will work locally for sufficiently small $\|x_\delta(0)\|$, $\|\hat{x}_\delta(0)\|$, and $\|u_\delta(t)\|$

Feedback Control:

$$\dot{\hat{x}}_\delta = A\hat{x}_\delta + Bu_\delta + H(y_\delta - C\hat{x}_\delta)$$

$$u_\delta = -K\hat{x}_\delta, \qquad u = u_{ss} - K\hat{x}_\delta$$

Verify that the closed-loop system has an equilibrium point at

$$x = x_{ss}, \qquad \tilde{x} = 0$$

and linearization at the equilibrium point yields

$$\begin{bmatrix} \dot{x}_\delta \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} (A - BK) & BK \\ 0 & (A - HC) \end{bmatrix} \begin{bmatrix} x_\delta \\ \tilde{x} \end{bmatrix}$$

Which theorem would justify this controller locally?

Nonlinear Observer via Linearization

$$\dot{x} = f(x, u), \qquad y = h(x)$$

$$0 = f(x_{ss}, u_{ss}), \qquad y_{ss} = h(x_{ss})$$

$$\dot{\hat{x}} = f(\hat{x}, u) + H[y - h(\hat{x})]$$

Equilibrium Point: $x = x_{ss}$, $u = u_{ss}$, $\hat{x} = x_{ss}$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = g(x, \tilde{x}, u), \qquad g(x_{ss}, 0, u_{ss}) = 0$$

Verify that linearization at the equilibrium point yields

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

Investigate the design of H and the use in feedback control

Extended Kalman Filter (EKF)

$$\dot{x} = f(x, u) + w, \quad y = h(x, u) + v$$

$$\dot{\hat{x}} = f(\hat{x}, u) + H(t)[y - h(\hat{x}, u)]$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = f(x, u) + w - f(\hat{x}, u) - H[h(x, u) + v - h(\hat{x}, u)]$$

Substitute $x = \hat{x} + \tilde{x}$ and expand the RHS in a Taylor series about $\tilde{x} = 0$

$$\dot{\tilde{x}} = [A(t) - H(t)C(t)]\tilde{x} + \eta(\tilde{x}, t) + \xi(t)$$

$$A(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)), \quad C(t) = \frac{\partial h}{\partial x}(\hat{x}(t), u(t))$$

$$\eta(0, t) = 0, \quad \xi(t) = w(t) - H(t)v(t)$$

Assuming that $x(t)$, $u(t)$, $w(t)$, $v(t)$, and $H(t)$ are bounded and f and h are twice continuously differentiable, show that

$$\|\eta(\tilde{x}, t)\| \le k_1 \|\tilde{x}\|^2, \quad \|\xi(t)\| \le k_2$$

Hint:

$$\begin{aligned} & f(x, u) - f(\hat{x}, u) - \frac{\partial f}{\partial x}(\hat{x}, u)\tilde{x} \\ & = \int_0^1 \frac{\partial f}{\partial x}(\sigma\tilde{x} + \hat{x}, u)\, d\sigma \tilde{x} - \frac{\partial f}{\partial x}(\hat{x}, u)\tilde{x} \quad \text{(Exercise 3.23)} \\ & = \int_0^1 \left[\frac{\partial f}{\partial x}(\sigma\tilde{x} + \hat{x}, u) - \frac{\partial f}{\partial x}(\hat{x}, u)\right]\, d\sigma\, \tilde{x} \end{aligned}$$

Kalman Filter Design: Let $Q(t)$ and $R(t)$ be symmetric positive definite matrices that satisfy

$$0 < q_1 I \le Q(t) \le q_2 I, \qquad 0 < r_1 I \le R(t) \le r_2 I$$

Let $P(t)$ be the solution of the Riccati equation

$$\dot{P} = AP + PA^T + Q - PC^T R^{-1} CP, \quad P(t_0) = P_0 > 0$$

If $(A(t), C(t)$ is uniformly observable, then $P(t)$ exists for all $t \ge t_0$ and satisfies

$$0 < p_1 I \le P(t) \le p_2 I \quad \Rightarrow \quad 0 < p_3 I \le P^{-1}(t) \le p_4 I$$

See a texbook on optimal control or optimal estimation

$$H(t) = P(t)C(t)^T R^{-1}(t)$$

- Compute $\boldsymbol{A}(\boldsymbol{t})$ and $\boldsymbol{C}(\boldsymbol{t})$

$$\boldsymbol{A}(\boldsymbol{t}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t})), \quad \boldsymbol{C}(\boldsymbol{t}) = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}}(\boldsymbol{t}), \boldsymbol{u}(\boldsymbol{t}))$$

- Solve the Riccati equation
- Compute $\boldsymbol{H}(\boldsymbol{t})$

$$\boldsymbol{H}(\boldsymbol{t}) = \boldsymbol{P}(\boldsymbol{t})\boldsymbol{C}(\boldsymbol{t})^{\boldsymbol{T}}\boldsymbol{R}^{-1}(\boldsymbol{t})$$

Remark: The Riccati equation and the observer equation have to be solved simultaneously in real time because $\boldsymbol{A}(\boldsymbol{t})$ and $\boldsymbol{C}(\boldsymbol{t})$ depend on $\hat{\boldsymbol{x}}(\boldsymbol{t})$ and $\boldsymbol{u}(\boldsymbol{t})$

Lemma: The origin of

$$\dot{\tilde{x}} = [A(t) - H(t)C(t)]\tilde{x} + \eta(\tilde{x}, t)$$

is exponentially stable and the solutions of

$$\dot{\tilde{x}} = [A(t) - H(t)C(t)]\tilde{x} + \eta(\tilde{x}, t) + \xi(t)$$

are uniformly ultimately bounded by an ultimate bound proportional to k_2

Proof:

$$V = \tilde{x}^T P^{-1} \tilde{x}$$

$$\dot{V} = \tilde{x}^T P^{-1} \dot{\tilde{x}} + \dot{\tilde{x}}^T P^{-1} \tilde{x} + \tilde{x}^T \frac{d}{dt} P^{-1} \tilde{x}$$

$$\frac{d}{dt}P^{-1} = -P^{-1}\dot{P}P^{-1}$$

$$\begin{aligned}
\dot{V} &= \tilde{x}^T P^{-1}(A - PC^T R^{-1} C)\tilde{x} \\
&\quad + \tilde{x}^T (A^T - C^T R^{-1} C P) P^{-1} \tilde{x} \\
&\quad - \tilde{x}^T P^{-1} \dot{P} P^{-1} \tilde{x} + 2\tilde{x}^T P^{-1}(\eta + \xi)
\end{aligned}$$

$$\begin{aligned}
\dot{V} &= \tilde{x}^T P^{-1}(AP + PA^T - PC^T R^{-1} CP - \dot{P}) P^{-1} \tilde{x} \\
&\quad - \tilde{x}^T C^T R^{-1} C \tilde{x} + 2\tilde{x}^T P^{-1}(\eta + \xi) \\
&= -\tilde{x}^T (P^{-1} Q P^{-1} + C^T R^{-1} C)\tilde{x} + 2\tilde{x}^T P^{-1}(\eta + \xi) \\
&\leq -c_1 \|\tilde{x}\|^2 + c_2 \|\tilde{x}\|^3 + c_3 \|\tilde{x}\| \qquad (c_3 \propto k_2)
\end{aligned}$$

Stochastic Interpretation: When $w(t)$ and $v(t)$ are

- zero-mean, white noise stochastic processes,
- uncorrelated, i.e., $E\{w(t)v^T(\tau)\} = 0,\ \forall t, \tau$, and
- $E\{w(t)w^T(\tau)\} = Q(t)\delta(t-\tau)$
  $E\{v(t)v^T(\tau)\} = R(t)\delta(t-\tau)$

then $\hat{x}(t)$ is an approximation of the minimum variance estimate that minimizes

$$E\left\{[y(t) - h(\hat{x}(t), u(t))]^T [y(t) - h(\hat{x}(t), u(t))]\right\}$$

and $P(t)$ is an approximation of the covariance matrix

$$E\left\{[\hat{x}(t) - x(t)][\hat{x}(t) - x(t)]^T\right\}$$

Feedback Control: What can you say about the closed-loop system when $\hat{x}$ is used in feedback control?