Nonlinear Systems and Control
Lecture # 36

Tracking

Equilibrium-to-Equilibrium Transition

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= \underbrace{L_f^\rho h(x)}_{f_b(\eta,\xi)} + \underbrace{L_g L_f^{\rho-1} h(x)}_{g_b(\eta,\xi)}\, u \\
y &= \xi_1
\end{aligned}
$$

Equilibrium point:

$$
\begin{aligned}
0 &= f_0(\bar{\eta}, \bar{\xi}) \\
0 &= \bar{\xi}_{i+1}, \quad 1 \le i \le \rho - 1 \\
0 &= f_b(\bar{\eta}, \bar{\xi}) + g_b(\bar{\eta}, \bar{\xi})\bar{u} \\
\bar{y} &= \bar{\xi}_1
\end{aligned}
$$

$$\bar{\xi}_1 = \bar{y}, \quad \bar{\xi}_i = 0 \quad \text{for } 2 \le i \le \rho - 1$$

$$0 = f_0(\bar{\eta}, \bar{y}, 0, \cdots, 0), \quad \bar{u} = -\frac{f_b(\bar{\eta}, \bar{y}, 0, \cdots, 0)}{g_b(\bar{\eta}, \bar{y}, 0, \cdots, 0)}$$

Assume $f_0(\bar{\eta}, \bar{y}, 0, \cdots, 0)$ has a unique solution $\bar{\eta}$ in the domain of interest

$$\bar{\eta} = \phi_\eta(\bar{y}), \quad \bar{u} = \phi_u(\bar{y})$$

By assumption (and without loss of generality)

$$\phi_\eta(0) = 0, \quad \phi_u(0) = 0$$

Goal: Move the system from equilibrium at $y = 0$ to equilibrium at $y = \bar{y}$, either asymptotically or over a finite time period

First Approach: Apply a step command

$$y^*(t) = \begin{cases} 0, & \text{for } t < 0 \\ \bar{y} & \text{for } t \geq 0 \end{cases}$$

Is this allowed ?

$$\text{Take} \quad r = \bar{y}, \quad \text{for } t \geq 0$$

$$r^{(i)} = 0 \ \text{ for } i \geq 2$$

$$\eta(0) = 0, \quad e_1(0) = -\bar{y}, \quad e_i(0) = 0 \text{ for } i \geq 2$$

The shape of the transient response depends on the solution of

$$\dot{e} = (A_c - B_c K)e$$

in feedback linearization
or the solution of

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_{\rho-1} \end{bmatrix} = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -k_1 & & & -k_{\rho-1} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{\rho-1} \end{bmatrix} + \begin{bmatrix} \\ \\ 1 \end{bmatrix} s$$

in sliding mode control
What is the impact of the reaching phase?

Second Approach: Take $r(t)$ as the zero-state response of a Hurwitz transfer function driven by $y*$

Typical Choice:

$$\frac{a_\rho}{s^\rho + a_1 s^{\rho-1} + \cdots + a_{\rho-1}s + a_\rho}$$

Choose the parameters a_1 to a_ρ to shape the response of r

$$r(0) = 0 \;\Rightarrow\; e_1(0) = 0 \;\Rightarrow\; e(0) = 0$$

Feedback Linearization:

Sliding Mode Control:

The derivatives of r are generated by the pre-filter

$$\begin{aligned} \dot{z} &= \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_\rho & & & -a_1 \end{bmatrix} z + \begin{bmatrix} \\ \\ \\ a_\rho \end{bmatrix} y^* \\ r &= \begin{bmatrix} 1 & & & \end{bmatrix} z \end{aligned}$$

$$r = z_1, \quad \dot{r} = z_2, \ldots\ldots \quad r^{(\rho-1)} = z_\rho$$

$$r^{(\rho)} = -\sum_{i=1}^{\rho} a_{\rho-i+1} z_i + a_\rho y^*$$

Does $r(t)$ satisfy the assumptions imposed last lecture?

Example 13.22

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -10\sin x_1 - x_2 + 10u, \quad y = x_1$$

Move the pendulum from equilibrium at $x_1 = 0$ to equilibrium at $x_1 = \frac{\pi}{2}$

$$Constraint: \quad |u(t)| \leq 2$$

$$y^* = \begin{cases} 0, & \text{for } t < 0 \\ \frac{\pi}{2} & \text{for } t \geq 0 \end{cases}$$

Pre-Filter:

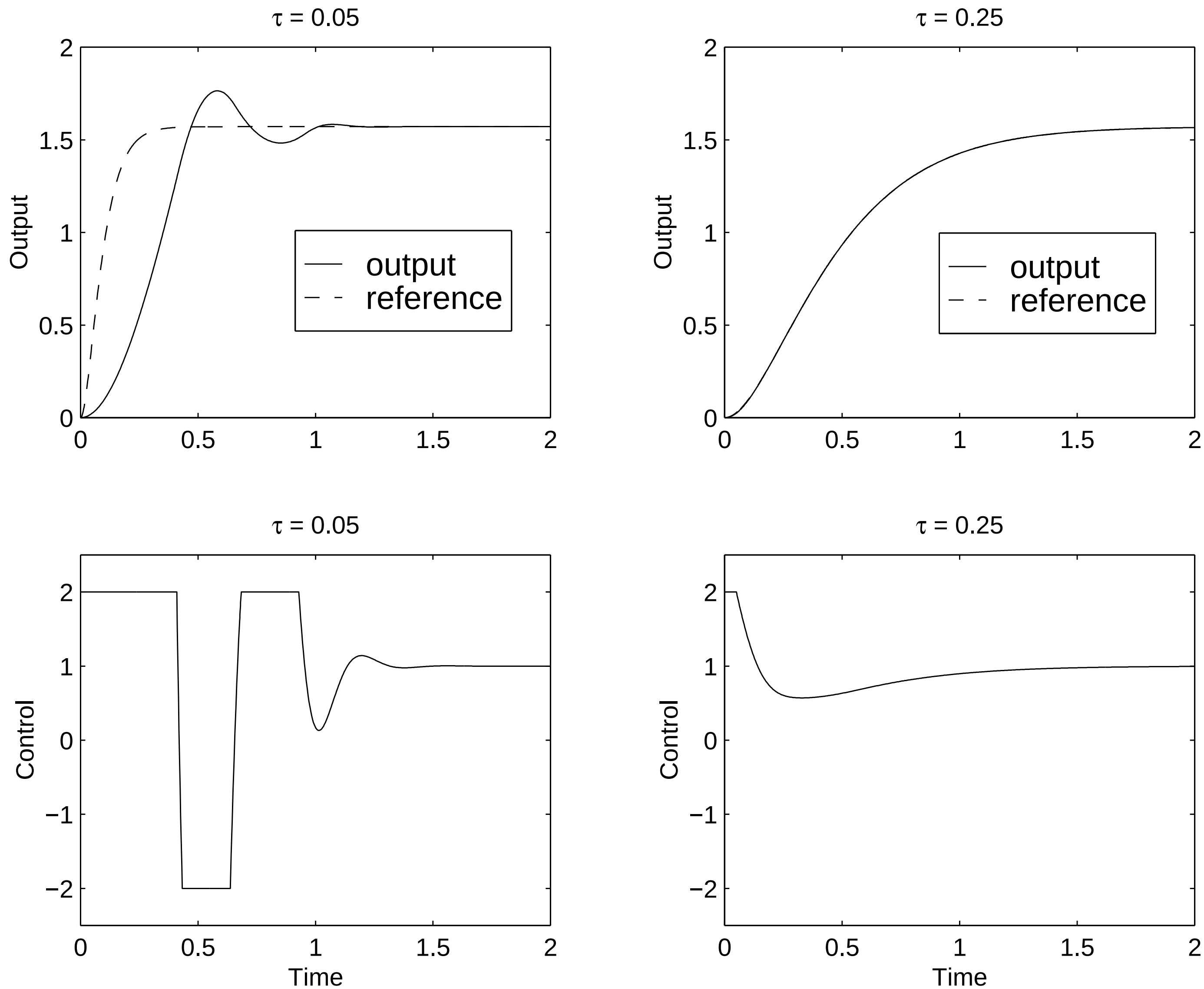
$$\frac{1}{(\tau s + 1)^2}$$

$$
\begin{aligned}
\dot{z} &= \begin{bmatrix} 0 & 1 \\ \frac{-1}{\tau^2} & \frac{-2}{\tau} \end{bmatrix} z + \begin{bmatrix} 0 \\ \frac{1}{\tau^2} \end{bmatrix} y^* \\
r &= \begin{bmatrix} 1 & 0 \end{bmatrix} z
\end{aligned}
$$

$$
r = z_1, \quad \dot{r} = z_2, \quad \ddot{r} = \frac{1}{\tau^2}(y^* - r) - \frac{2}{\tau} z_2
$$

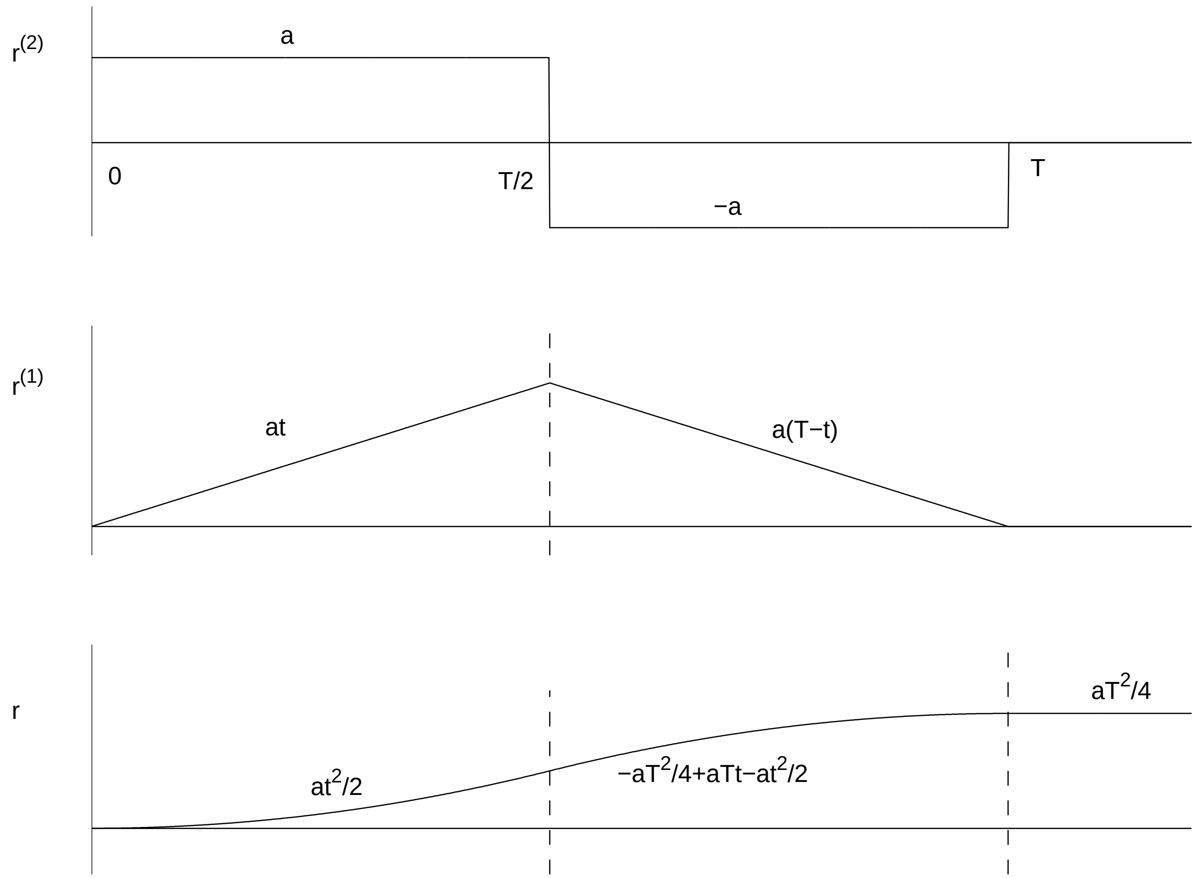
$$
r(t) = \frac{\pi}{2}\left[1 - e^{-\frac{t}{\tau}}\left(1 + \frac{t}{\tau}\right)\right]
$$

$$
u = 0.1(10 \sin x_1 + x_2 + \ddot{r} - k_1 e_1 - k_2 e_2)
$$


τ = 0.05
τ = 0.25
Output
output
reference
Control
Time

Third Approach: Plan a trajectory $(r(t), \dot{r}(t), \ldots, r^{(\rho)}(t))$ to move from $(0, 0, \ldots, 0)$ to $(\bar{y}, 0, \ldots, 0)$ in finite time T

Example: $\rho = 2$



$$r(t) = \begin{cases} \frac{at^2}{2} & \text{for } 0 \leq t \leq \frac{T}{2} \\ -\frac{aT^2}{4} + aTt - \frac{at^2}{2} & \text{for } \frac{T}{2} \leq t \leq T \\ \frac{aT^2}{4} & \text{for } t \geq T \end{cases}$$

$$a = \frac{4\bar{y}}{T^2} \quad \Rightarrow \quad r(t) = \bar{y} \quad \text{for } t \geq T$$