Nonlinear Systems and Control
Lecture # 30

Stabilization

Control Lyapunov Functions

$$\dot{x} = f(x) + g(x)u, \qquad f(0) = 0, \; x \in R^n, \; u \in R$$

Suppose there is a continuous stabilizing state feedback control $u = \psi(x)$ such that the origin of

$$\dot{x} = f(x) + g(x)\psi(x)$$

is asymptotically stable

By the converse Lyapunov theorem, there is $V(x)$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)\psi(x)] < 0, \quad \forall \, x \in D, \; x \neq 0$$

If $u = \psi(x)$ is globally stabilizing, then $D = R^n$ and $V(x)$ is radially unbounded

$$\frac{\partial V}{\partial x}[f(x) + g(x)\psi(x)] < 0, \quad \forall\, x \in D,\ x \neq 0$$

$$\frac{\partial V}{\partial x} g(x) = 0 \text{ for } x \in D,\ x \neq 0 \ \Rightarrow\ \frac{\partial V}{\partial x} f(x) < 0$$

Since $\psi(x)$ is continuous and $\psi(0) = 0$, given any $\varepsilon > 0$, $\exists\ \delta > 0$ such that if $x \neq 0$ and $\|x\| < \delta$, there is u with $\|u\| < \varepsilon$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)u] < 0$$

Small Control Property

Definition: A continuously differentiable positive definite function $V(x)$ is a *Control Lyapunov Function* (CLF) for the system $\dot{x} = f(x) + g(x)u$ if

- $$\frac{\partial V}{\partial x} g(x) = 0 \text{ for } x \in D,\ x \neq 0 \ \Rightarrow\ \frac{\partial V}{\partial x} f(x) < 0 \qquad (*)$$

- it satisfies the small control property

It is a *Global Control Lyapunov Function* if it is radially unbounded and $(*)$ holds with $D = R^n$

The system $\dot{x} = f(x) + g(x)u$ is stabilizable by a continuous state feedback control <u>only if</u> it has a CLF

Is it sufficient?

Theorem: Let $V(x)$ be a CLF for $\dot{x} = f(x) + g(x)u$, then origin is stabilizable by $u = \psi(x)$, where

$$\psi(x) = \begin{cases} -\dfrac{\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}}{\left(\frac{\partial V}{\partial x} g\right)}, & \text{if } \dfrac{\partial V}{\partial x} g \neq 0 \\ 0, & \text{if} \dfrac{\partial V}{\partial x} g = 0 \end{cases}$$

The function $\psi(x)$ is continuous for all $x \in D_0$ (a neighborhood of the origin) including $x = 0$. If f and g are smooth, then ψ is smooth for $x \neq 0$. If V is a global CLF, then the control $u = \psi(x)$ is globally stabilizing

Sontag's Formula

Proof: For properties of ψ, see Section 9.4 of [88]

$$\frac{\partial V}{\partial x}[f(x) + g(x)\psi(x)]$$

If $\dfrac{\partial V}{\partial x} g(x) = 0, \quad \dot{V} = \dfrac{\partial V}{\partial x} f(x) < 0$ for $x \neq 0$

If $\dfrac{\partial V}{\partial x} g(x) \neq 0$

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} f - \left[\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} \right] \\ &= -\sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} < 0 \text{ for } x \neq 0 \end{aligned}$$

How can we find a CLF?

If we know of any stabilizing control with a corresponding Lyapunov function V, then V is a CLF

- Feedback Linearization

$$\dot{x} = f(x) + G(x)u, \quad z = T(x), \quad \dot{z} = (A - BK)z$$

$$P(A - BK) + (A - BK)^T P = -Q, \;\; Q = Q^T > 0$$

$$V = z^T P z = T^T(x) P T(x) \text{ is a CLF}$$

- Backstepping

Example:

$$\dot{x} = ax - bx^3 + u, \qquad a, b > 0$$

Feedback Linearization:

$$u = -ax + bx^3 - kx \qquad (k > 0)$$

$$\dot{x} = -kx$$

$V(x) = \frac{1}{2}x^2$ is a CLF

$$\frac{\partial V}{\partial x} g = x, \qquad \frac{\partial V}{\partial x} f = x(ax - bx^3)$$

$$
\begin{aligned}
& - \frac{\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}}{\left(\frac{\partial V}{\partial x} g\right)} \\
= \quad & - \frac{x(ax - bx^3) + \sqrt{x^2(ax - bx^3)^2 + x^4}}{x} \\
= \quad & -ax + bx^3 - x\sqrt{(a - bx^2)^2 + 1}
\end{aligned}
$$

$$
\psi(x) = -ax + bx^3 - x\sqrt{(a - bx^2)^2 + 1}
$$

Compare with

$$
-ax + bx^3 - kx
$$

Method	Expression
FL-u	$-ax + bx^3 - kx$
FL-CLS	$\dot{x} = -kx$
CLF-u	$-ax + bx^3 - x\sqrt{(a - bx^2)^2 + 1}$
CLF-CLS	$-x\sqrt{(a - bx^2)^2 + 1}$

Method	Small $\lvert x \rvert$	Large $\lvert x \rvert$
FL-u	$(-a + k)x$	bx^3
FL-CLS	$\dot{x} = -kx$	$\dot{x} = -kx$
CLF-u	$-(a + \sqrt{a^2 + 1})x$	$-ax$
CLF-CLS	$-\sqrt{a^2 + 1}x$	$-bx^3$

Lemma: Let $V(x)$ be a CLF for $\dot{x} = f(x) + g(x)u$ and suppose $\frac{\partial V}{\partial x}(0) = 0$. Then, Sontag's formula has a gain margin $[\frac{1}{2}, \infty)$; that is, $u = k\psi(x)$ is stabilizing for all $k \geq \frac{1}{2}$

Proof: Let

$$q(x) = \tfrac{1}{2}\left[-\frac{\partial V}{\partial x}f + \sqrt{\left(\frac{\partial V}{\partial x}f\right)^2 + \left(\frac{\partial V}{\partial x}g\right)^4}\right]$$

$$q(0) = 0, \quad \frac{\partial V}{\partial x}g \neq 0 \;\Rightarrow\; q > 0$$

$$\frac{\partial V}{\partial x}g = 0 \;\Rightarrow\; q = -\frac{\partial V}{\partial x}f > 0 \text{ for } x \neq 0$$

$q(x)$ is positive definite

$$u = k\psi(x) \;\; \Rightarrow \;\; \dot{x} = f(x) + g(x)k\psi(x)$$

$$\dot{V} = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} g k \psi$$

$$\frac{\partial V}{\partial x} g = 0 \;\; \Rightarrow \;\; \dot{V} = \frac{\partial V}{\partial x} f < 0 \text{ for } x \neq 0$$

$$\frac{\partial V}{\partial x} g \neq 0, \quad \dot{V} = -q + q + \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} g k \psi$$

$$\begin{aligned} & q + \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} g k \psi \\ = \;\; & -\left(k - \tfrac{1}{2}\right) \left[\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} \right] \end{aligned}$$