Nonlinear Systems and Control
Lecture # 27

Stabilization

Partial Feedback Linearization

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u \qquad [f(0) = 0]$$

Suppose there is a change of variables

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$$

defined for all $x \in D \subset R^n$, that transforms the system into

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A\xi + B\gamma(x)[u - \alpha(x)] \end{aligned}$$

(A, B) is controllable and $\gamma(x)$ is nonsingular for all $x \in D$

$$u = \alpha(x) + \gamma^{-1}(x)v$$

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = A\xi + Bv$$

Suppose the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable

$$v = -K\xi, \quad \text{where } (A - BK) \text{ is Hurwitz}$$

Lemma 13.1: The origin of

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi$$

is asymptotically stable if the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable

Proof: $$V(\eta, \xi) = V_1(\eta) + k\sqrt{\xi^T P \xi}$$

If the origin of $\dot{\eta} = f_0(\eta, 0)$ is globally asymptotically stable, will the origin of

$$\dot{\eta} = f_0(\eta, \xi), \qquad\qquad \dot{\xi} = (A - BK)\xi$$

be globally asymptotically stable? In general No

Example

$$\dot{\eta} = -\eta + \eta^2\xi, \qquad\qquad \dot{\xi} = v$$

The origin of $\dot{\eta} = -\eta$ is globally exponentially stable, but the origin of

$$\dot{\eta} = -\eta + \eta^2\xi, \qquad\qquad \dot{\xi} = -k\xi, \quad k > 0$$

is not globally asymptotically stable. The region of attraction is $\{\eta\xi < 1 + k\}$

Example

$$\dot{\eta} = -\tfrac{1}{2}(1+\xi_2)\eta^3, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = v$$

The origin of $\dot{\eta} = -\frac{1}{2}\eta^3$ is globally asymptotically stable

$$v = -k^2\xi_1 - 2k\xi_2 \stackrel{\text{def}}{=} -K\xi \quad \Rightarrow \quad A - BK = \begin{bmatrix} 0 & 1 \\ -k^2 & -2k \end{bmatrix}$$

The eigenvalues of $(A - BK)$ are $-k$ and $-k$

$$e^{(A-BK)t} = \begin{bmatrix} (1+kt)e^{-kt} & te^{-kt} \\ \\ -k^2te^{-kt} & (1-kt)e^{-kt} \end{bmatrix}$$

Peaking Phenomenon:

$$\max_t \{k^2 t e^{-kt}\} = \frac{k}{e} \rightarrow \infty \text{ as } k \rightarrow \infty$$

$$\xi_1(0) = 1, \; \xi_2(0) = 0 \quad \Rightarrow \quad \xi_2(t) = -k^2 t e^{-kt}$$

$$\dot{\eta} = -\tfrac{1}{2}\left(1 - k^2 t e^{-kt}\right)\eta^3, \qquad \eta(0) = \eta_0$$

$$\eta^2(t) = \frac{\eta_0^2}{1 + \eta_0^2[t + (1 + kt)e^{-kt} - 1]}$$

If $\eta_0^2 > 1$, the system will have a finite escape time if k is chosen large enough

Lemma 13.2: The origin of

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi$$

is globally asymptotically stable if the system $\dot{\eta} = f_0(\eta, \xi)$ is input-to-state stable

Proof: Use
Lemma 4.7: If $\dot{x}_1 = f_1(x_1, x_2)$ is ISS and the origin of $\dot{x}_2 = f_2(x_2)$ is globally asymptotically stable, then the origin of

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_2)$$

is globally asymptotically stable

$$u = \alpha(x) - \gamma^{-1}(x) K T_2(x)$$

What is the effect of uncertainty in α, γ, and T_2?

Let $\hat{\alpha}(x)$, $\hat{\gamma}(x)$, and $\hat{T}_2(x)$ be nominal models of $\alpha(x)$, $\gamma(x)$, and $T_2(x)$

$$u = \hat{\alpha}(x) - \hat{\gamma}^{-1}(x) K \hat{T}_2(x)$$

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi + B\delta(z)$$

$$\delta = \gamma[\hat{\alpha} - \alpha + \gamma^{-1} K T_2 - \hat{\gamma}^{-1} K \hat{T}_2]$$

Lemma 13.4

- If $\|\delta(z)\| \leq \varepsilon$ for all z and $\dot{\eta} = f_0(\eta, \xi)$ is input-to-state stable, then the state z is globally ultimately bounded by a class $\mathcal{K}$ function of ε
- If $\|\delta(z)\| \leq k\|z\|$ in some neighborhood of $z = 0$, with sufficiently small k, and the origin of $\dot{\eta} = f_0(\eta, 0)$ is exponentially stable, then $z = 0$ is an exponentially stable equilibrium point of the system

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi + B\delta(z)$$

Proof–First Part: As in Lemma 13.3

$$\|\xi(t)\|_{\leq} c\varepsilon, \quad \forall\, t \geq t_0$$

$$\|\eta(t)\| \leq \beta_0(\|\eta(t_0)\|, t - t_0) + \gamma_0(\sup_{t \geq t_0} \|\xi(t)\|)$$

$$\|\eta(t)\| \leq \beta_0(\|\eta(t_0)\|, t - t_0) + \gamma_0(c\varepsilon)$$

Proof–Second Part:

$$c_1\|\eta\|^2 \leq V_1(\eta) \leq c_2\|\eta\|^2$$

$$\frac{\partial V_1}{\partial \eta} f_0(\eta, 0) \leq -c_3\|\eta\|^2$$

$$\left\|\frac{\partial V_1}{\partial \eta}\right\| \leq c_4\|\eta\|$$

$$V(z) = bV_1(\eta) + \xi^T P \xi$$

$$\dot{V} \leq - \begin{bmatrix} \|\eta\| \\ \|\xi\| \end{bmatrix}^T Q \begin{bmatrix} \|\eta\| \\ \|\xi\| \end{bmatrix}$$

$$Q = \begin{bmatrix} bc_3 & -(k\|PB\| + bc_4 L/2) \\ -(k\|PB\| + bc_4 L/2) & 1 - 2k\|PB\| \end{bmatrix}$$

$$b = k$$

Q is positive definite for sufficiently small k