Nonlinear Systems and Control
Lecture # 26

Stabilization

Feedback Lineaization

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u$$

$$f(0) = 0, \quad x \in R^n, \quad u \in R^m$$

Suppose there is a change of variables $z = T(x)$, defined for all $x \in D \subset R^n$, that transforms the system into the controller form

$$\dot{z} = Az + B\gamma(x)[u - \alpha(x)]$$

where (A, B) is controllable and $\gamma(x)$ is nonsingular for all $x \in D$

$$u = \alpha(x) + \gamma^{-1}(x)v \quad \Rightarrow \quad \dot{z} = Az + Bv$$

$$\boldsymbol{v} = -\boldsymbol{K}\boldsymbol{z}$$

Design $\boldsymbol{K}$ such that $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})$ is Hurwitz

The origin $\boldsymbol{z} = \boldsymbol{0}$ of the closed-loop system

$$\dot{\boldsymbol{z}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{z}$$

is globally exponentially stable

$$\boldsymbol{u} = \boldsymbol{\alpha}(\boldsymbol{x}) - \gamma^{-1}(\boldsymbol{x})\boldsymbol{K}\boldsymbol{T}(\boldsymbol{x})$$

Closed-loop system in the $\boldsymbol{x}$-coordinates:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\left[\boldsymbol{\alpha}(\boldsymbol{x}) - \gamma^{-1}(\boldsymbol{x})\boldsymbol{K}\boldsymbol{T}(\boldsymbol{x})\right]$$

What can we say about the stability of $\boldsymbol{x} = \mathbf{0}$ as an equilibrium point of

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x}) \left[\boldsymbol{\alpha}(\boldsymbol{x}) - \gamma^{-1}(\boldsymbol{x})\boldsymbol{K}\boldsymbol{T}(\boldsymbol{x})\right]$$

$\boldsymbol{x} = \mathbf{0}$ is asymptotically stable because $\boldsymbol{T}(\boldsymbol{x})$ is a diffeomorphism. Show it!

Is $\boldsymbol{x} = \mathbf{0}$ globally asymptotically stable? In general No

It is globally asymptotically stable if $\boldsymbol{T}(\boldsymbol{x})$ is a global diffeomorphism (See page 508)

What information do we need to implement the control

$$\boldsymbol{u} = \boldsymbol{\alpha}(\boldsymbol{x}) - \boldsymbol{\gamma}^{-1}(\boldsymbol{x})\boldsymbol{K}\boldsymbol{T}(\boldsymbol{x}) \text{ ?}$$

What is the effect of uncertainty in $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, and $\boldsymbol{T}$?

Let $\hat{\boldsymbol{\alpha}}(\boldsymbol{x})$, $\hat{\boldsymbol{\gamma}}(\boldsymbol{x})$, and $\hat{\boldsymbol{T}}(\boldsymbol{x})$ be nominal models of $\boldsymbol{\alpha}(\boldsymbol{x})$, $\boldsymbol{\gamma}(\boldsymbol{x})$, and $\boldsymbol{T}(\boldsymbol{x})$

$$\boldsymbol{u} = \hat{\boldsymbol{\alpha}}(\boldsymbol{x}) - \hat{\boldsymbol{\gamma}}^{-1}(\boldsymbol{x})\boldsymbol{K}\hat{\boldsymbol{T}}(\boldsymbol{x})$$

Closed-loop system:

$$\dot{\boldsymbol{z}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{z} + \boldsymbol{B}\boldsymbol{\delta}(\boldsymbol{z})$$

$$\boldsymbol{\delta} = \boldsymbol{\gamma}[\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} + \boldsymbol{\gamma}^{-1}\boldsymbol{K}\boldsymbol{T} - \hat{\boldsymbol{\gamma}}^{-1}\boldsymbol{K}\hat{\boldsymbol{T}}]$$

$$\dot{z} = (A - BK)z + B\delta(z) \qquad (*)$$

$$V(z) = z^T P z, \quad P(A - BK) + (A - BK)^T P = -I$$

Lemma 13.3

- If $\|\delta(z)\| \leq k\|z\|$ for all z, where

$$0 \leq k < \frac{1}{2\|PB\|}$$

  then the origin of (*) is globally exponentially stable

- If $\|\delta(z)\| \leq k\|z\| + \varepsilon$ for all z, then the state z is globally ultimately bounded by εc for some $c > 0$

Example (Pendulum Equation):

$$\ddot{\theta} = -a \sin\theta - b\dot{\theta} + cT$$

$$x_1 = \theta - \delta_1, \quad x_2 = \dot{\theta}, \quad u = T - T_{\mathrm{ss}} = T - \frac{a}{c}\sin\delta_1$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a[\sin(x_1 + \delta_1) - \sin\delta_1] - bx_2 + cu \end{aligned}$$

$$u = \frac{1}{c}\{a[\sin(x_1 + \delta_1) - \sin\delta_1] - k_1 x_1 - k_2 x_2\}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2 + b) \end{bmatrix}$$ is Hurwitz

$$T = u + \frac{a}{c}\sin\delta_1 = \frac{1}{c}\left[a\sin(x_1+\delta_1) - k_1x_1 - k_2x_2\right]$$

Let $\hat{a}$ and $\hat{c}$ be nominal models of a and c

$$T = \frac{1}{\hat{c}}\left[\hat{a}\sin(x_1+\delta_1) - k_1x_1 - k_2x_2\right]$$

$$\dot{x} = (A - BK)x + B\delta(x)$$

$$\delta(x) = \left(\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right)\sin(x_1+\delta_1) - \left(\frac{c-\hat{c}}{\hat{c}}\right)(k_1x_1 + k_2x_2)$$

$$\delta(x) = \left(\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right)\sin(x_1 + \delta_1) - \left(\frac{c - \hat{c}}{\hat{c}}\right)(k_1 x_1 + k_2 x_2)$$

$$|\delta(x)| \leq k\|x\| + \varepsilon$$

$$k = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| + \left|\frac{c - \hat{c}}{\hat{c}}\right| \sqrt{k_1^2 + k_2^2}, \quad \varepsilon = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| \, |\sin\delta_1|$$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \quad PB = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}$$

$$k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}}$$

$$\sin\delta_1 = 0 \;\Rightarrow\; \varepsilon = 0$$

Is feedback linearization a good idea?

Example

$$\dot{x} = ax - bx^3 + u, \quad a, b > 0$$

$$u = -(k + a)x + bx^3, \ k > 0, \ \Rightarrow \ \dot{x} = -kx$$

$-bx^3$ is a damping term. Why cancel it?

$$u = -(k + a)x, \ k > 0, \ \Rightarrow \ \dot{x} = -kx - bx^3$$

Which design is better?

Example

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h(x_1) + u \\
& h(0) = 0 \text{ and } x_1 h(x_1) > 0, \ \forall\, x_1 \neq 0
\end{aligned}
$$

Feedback Linearization:

$$u = h(x_1) - (k_1 x_1 + k_2 x_2)$$

With $y = x_2$, the system is passive with

$$V = \int_0^{x_1} h(z)\, dz + \tfrac{1}{2} x_2^2$$

$$\dot{V} = h(x_1)\dot{x}_1 + x_2 \dot{x}_2 = yu$$

The control

$$u = -\sigma(x_2), \quad \sigma(0) = 0,\ x_2\sigma(x_2) > 0 \ \forall\, x_2 \neq 0$$

creates a feedback connection of two passive systems with storage function V

$$\dot{V} = -x_2\sigma(x_2)$$

$$x_2(t) \equiv 0 \ \Rightarrow \ \dot{x}_2(t) \equiv 0 \ \Rightarrow \ h(x_1(t)) \equiv 0 \ \Rightarrow \ x_1(t) \equiv 0$$

Asymptotic stability of the origin follows from the invariance principle

Which design is better? (Read Example 13.20)